Schubert Calculus Day 5: Equivariant Schubert Calculus

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What is Equivariant Cohomology?

When a space X admits an action by an algebraic group G, we can define invariants of X that take into account this symmetry. Concretely, instead of the cohomology ring $H^*(X)$ we also have the *equivariant cohomology ring* $H^*_G(X)$.

Definition

Let $\mathbb{E}G$ be a contractible space with a free (right) G-action. Consider the quotient space

$$\mathbb{E}G \times^G X = \mathbb{E}G \times X/(e \cdot g, x) \sim (e, g \cdot x).$$

Then the *equivariant cohomology* of X with respect to G is

$$H^*_G(X) = H^*(\mathbb{E}G \times^G X).$$

The choice of $\mathbb{E}G$ does not affect the equivariant ring.

Equivariant Cohomology

When $X = \{*\}$, $\mathbb{E}G \times^G \{*\} = \mathbb{B}G$ is the *classifying space* of G. It has fundamental group $\pi_1(\mathbb{B}G) = G$ and trivial higher homotopy groups. By general theory, $\mathbb{B}G$ is unique up to homotopy.

Example

Let $G = \mathbb{C}^*$. Then $\mathbb{E}G = \mathbb{C}^\infty - \{0\}$ is contractible and G acts freely on it, so $\mathbb{B}G = \mathbb{E}G/G = \mathbb{P}^\infty$. Hence

$$H^*_{\mathbb{C}^*}(\mathrm{pt}) = H^*\mathbb{P}^\infty \simeq \mathbb{Z}[t].$$

If X admits an action of a product $G \times H$, then we have $\mathbb{E}(G \times H) = \mathbb{E}G \times \mathbb{E}H$ and so

$$H^*_{G\times H}(X) = H^*_G(X) \otimes H^*_H(X).$$

In particular, we have $H^*_T(X) \simeq \mathbb{Z}[t_1, \ldots, t_n]$ where $T = (\mathbb{C}^*)^n$.

Equivariant Cohomology

Since $\mathbb{E}G$ and $\mathbb{B}G$ are often infinite-dimensional, it is better to work with finite-dimensional approximations to them.

Lemma

Let $\mathbb{E}_m$ be a connected space with a free right G-action and $H^i\mathbb{E}_m = 0$ for $0 < i < k(m)$ and some integer $k(m)$. Then there are natural isomorphisms

$$H^i(\mathbb{E}_m \times^G X) \simeq H^i(\mathbb{E} \times^G X) := H^i_G X \qquad \text{for } i < k(m).$$

Example

1. When $G = \mathbb{C}^*$, take $\mathbb{E}_m = \mathbb{C}^m - \{0\}$. Then $H^i\mathbb{E}_m = 0$ for $0 < i < 2m - 1$.
2. When $G = \mathrm{GL}(n)$, take $\mathbb{E}_m = \{m \times n \text{ matrices of rank } n\}$ for $m > n$. We can compute that $H^i\mathbb{E}_m = 0$ for $0 < i < 2(m - n)$. Moreover,

$$\mathbb{B}_m = \mathbb{E}_m / G = \mathrm{Gr}(n, \mathbb{C}^m).$$

Functorial Properties

Equivariant cohomology satisfies many of the same properties that (ordinary) cohomology does. For instance, it is functorial for *equivariant maps*.

Given a homomorphism $G \xrightarrow{\varphi} G'$ and a map $X \xrightarrow{f} X'$ such that $f(g \cdot x) = \varphi(g) \cdot f(x)$, we get a pullback map

$$f^* : H^*_{G'}X' \to H^*_G X.$$

When X is a nonsingular variety, any G-invariant subvariety Z of codimension d defines a *equivariant class*

$$[Z]^G = [\mathbb{E}_m \times^G Z] \in H^{2d}(\mathbb{E}_m \times^G X) = H^{2d}_G X,$$

for any approximation $\mathbb{E}_m$ with $m >> 0$.

Restriction & Localization

There are also two notions that are unique to the equivariant setting. The first is that *equivariant cohomology restricts to ordinary cohomology*. From the diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{E} \times^G X \\ \downarrow & & \downarrow \\ \text{pt} & \longrightarrow & \mathbb{B} \end{array},$$

we know that $H_G^* X$ is an algebra over $H_G^*(\text{pt})$. Under *nice conditions*, the pullback map

$$H_G^* X \to H^* X$$

will be surjective with kernel generated by the kernel of $H_G^*(\text{pt}) \to \mathbb{Z}$.

Restriction & Localization

The second notion is that *equivariant cohomology is determined at the fixed locus*. Assuming the G-action on X has finitely many fixed points X^G, under nice conditions the following map is injective

$$\iota^* : H_G^* X \to H_G^* X^G = \bigoplus_{x \in X^G} H_G^*(\mathrm{pt}),$$

where $\iota : X^G \to X$ is the inclusion.

Theorem

When $G = T$ and X is a nonsingular projective variety with finitely many fixed points, both notions hold. In other words, we have

$$H_T^* X \twoheadrightarrow H^* X \qquad \text{and} \qquad H_T^* X \hookrightarrow H_T^* X^T.$$

Torus Fixed Points

Recall that the Grassmannian $\mathrm{Gr}(k,n)$ can be defined as

$$\{n \times k \text{ matrices of rank } k\}/\,\mathrm{GL}(k),$$

which admits a left action of $T = (\mathbb{C}^*)^n$ on the left. In other words, T acts on $\mathbb{C}^n$ by scaling each basis vector. Thus, the k-dimensional subspaces of $\mathbb{C}^n$ that are T-invariant are the ones of the form

$$\langle e_{i_1}, \dots, e_{i_k} \rangle \qquad \text{for all} \qquad \{i_1, \dots, i_k\} \subset [n].$$

Corollary

$H_T^*(Gr(k,n))$ is an algebra over $H_T^*(pt) = \mathbb{Z}[t_1, \dots, t_n]$ and we have an embedding

$$H_T^*(Gr(k,n)) \hookrightarrow \bigoplus_{I \subset [n], |I|=k} \mathbb{Z}[t_1, \dots, t_n].$$

Equivariant Schubert Classes

Now, recall the Schubert stratification

$$\mathrm{Gr}(k,n) = \bigsqcup_{\lambda \subset (k^{n-k})} \Omega_\lambda^\circ = \bigsqcup_{I \subset [n], |I| = k} \Omega_I^\circ.$$

Here, the Schubert cell $\Omega_I^\circ = \{U \in \mathrm{Gr}(k,n) \mid \mathrm{position}(U) = I\}$ is fixed under T. This gives a well-defined *equivariant Schubert classes*

$$\widetilde{\sigma}_I = [\Omega_I]^T \in H_T^*(\mathrm{Gr}(k,n)).$$

Theorem

$H_T^*(Gr(k,n))$ is a free $\mathbb{Z}[t_1, \ldots, t_n]$-module generated by the equivariant Schubert classes $\widetilde{\sigma}_I$'s.

Schubert Representatives

Question

What are the images of the equivariant Schubert classes under the localization map

$$H_T^*(\mathrm{Gr}(k,n)) \hookrightarrow \bigoplus_{I \subset [n], |I|=k} \mathbb{Z}[t_1, \ldots, t_n] \text{ ?}$$

Denote by $(\widetilde{\sigma}_I)|_J$ the restriction of an equivariant Schubert class to the fixed point indexed by J.

Lemma

For every $I, J \subset [n]$ of length k, $(\widetilde{\sigma}_I)|_J \in \mathbb{N}[t_2 - t_1, t_3 - t_2, \ldots, t_n - t_{n-1}]$.

Schubert Representatives

The Schubert representatives $(\widetilde{\sigma}_I)|_J$ are uniquely determined by the following result.

Theorem

$\widetilde{\sigma}_I$ is the unique class in $H_T^*(Gr(k,n))$ such that

1. *its degree is $\#\{(i,j) : i \in I, j \notin I, j > i\}$,*
2. $$(\widetilde{\sigma}_I)|_I = \prod_{i \in I, j \notin I, j > i} (z_j - z_i),$$
3. *$(\widetilde{\sigma}_I)|_J = 0$ if $J \nleqslant I$.*

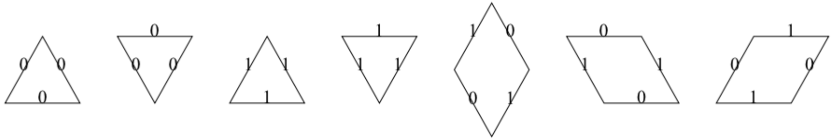
Here, the partial order $J = \{j_1, \ldots, j_k\} \leqslant I = \{i_1, \ldots, i_k\}$ is when $j_s \leqslant i_s$ for all $s = 1, \ldots, k$.

What is a Puzzle?

Introduced by Allen Knutson and Terence Tao in 2001, their *puzzles* are a class of objects counted by the Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$ and their equivariant counterpart $c_{\lambda\mu}^{\nu}(t)$ satisfying

$$\widetilde{\sigma}_\lambda \cdot \widetilde{\sigma}_\mu = \sum_\nu c_{\lambda\mu}^{\nu}(t)\widetilde{\sigma}_\nu.$$

For $\mathrm{Gr}(k, n)$, each puzzle is a way to tile a $n \times n$ equilateral triangle with these 1×1 pieces.



Ordinary Puzzles

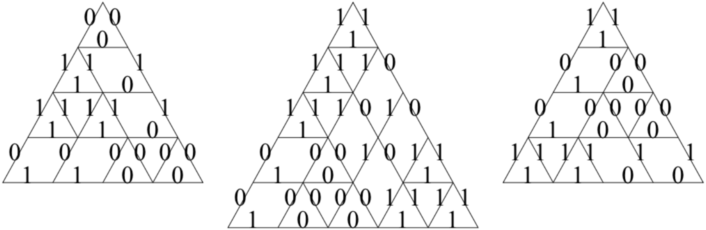


Figure: Some examples of puzzles (taken from the Knutson-Tao paper)

Theorem

The Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$ is the number of puzzles P whose three sides NW, NE, S represent λ, μ and ν respectively.

Equivariant Puzzles

For equivariant LR coefficients $c_{\lambda\mu}^{\nu}(t)$, we need to consider puzzles with one extra "equivariant" piece allowed.

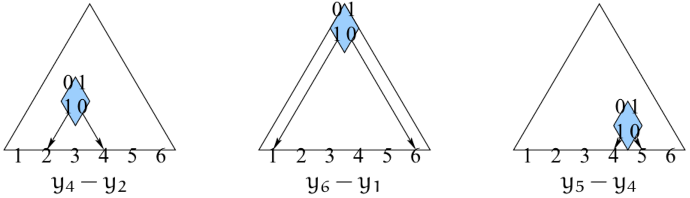


Figure: Equivariant piece and its weight

If a puzzle consists of multiple equivariant pieces, we multiply their weights to form the weight of the puzzle.

Equivariant Puzzles

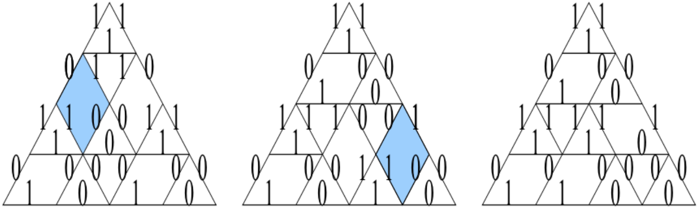


Figure: Equivariant puzzles for $\widetilde{\sigma}_{0101} \cdot \widetilde{\sigma}_{1010}$

Theorem

The equivariant LR coefficient $c^{\nu}_{\lambda\mu}(t)$ is the sum of the weight of the equivariant puzzles whose NW, NE and S sides are λ, μ and ν respectively.

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