Schubert Calculus Day 4: Schubert Calculus on the Flag Variety

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June 11, 2021

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What's next?

Now that we have studied Schubert calculus on the Grassmannian, how can we generalize?

1. We can study different cohomology theories: equivariant cohomology, K-theory, quantum cohomology, etc.
2. We can study cohomology of different varieties: flag variety, Grassmannians in other Dynkin types, affine Grassmannian, etc.

There are still a huge number of open questions for these generalizations. Today, we will explore option 2 by studying intersection theory on the flag variety.

Flag Variety

A (complete) flag for a n-dimensional vector space $V \simeq \mathbb{C}^n$ is a nested sequence of subspaces $\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$ with $\dim V_i = i$ for all i. The *(full) flag variety* $\mathrm{F}\ell(V) = \mathrm{F}\ell(n)$ is an object that parametrizes complete flags of V.

There are two ways to make this definition precise. The first is to define $\mathrm{F}\ell(n)$ as an *incidence relation* on $\mathrm{Gr}(1, n) \times \cdots \times \mathrm{Gr}(n-1, n)$:

$$\mathrm{F}\ell(n) = \left\{ (V_1, \ldots, V_{n-1}) \in \prod_{k=1}^{n-1} \mathrm{Gr}(k, n) \mid V_i \subset V_{i+1} \text{ for all } i \right\}.$$

We can show that $\mathrm{F}\ell(n)$ is cut out by quadratic equations in the Plücker coordinates of $\mathrm{Gr}(k, n)$'s, hence $\mathrm{F}\ell(n)$ is itself a projective variety.

Flag Variety

The second method is to note that we can choose a basis $v_1, \ldots, v_n$ of V so that $V_i = \langle v_1, \ldots, v_i \rangle$ for all i. Two bases determine the same flag if they are related by an lower-triangular matrix. Denote $G = \mathrm{GL}(n)$ and $B = \{\text{invertible lower-triangular matrices}\}$. Then

$$\mathrm{F}\ell(n) = G/B.$$

In particular, we can calculate $\dim \mathrm{F}\ell(n) = \binom{n}{2}$.

Both approaches also work for more general *partial flag variety*

$$\mathrm{F}\ell^{d_1,\ldots,d_r}(n) = \{\{0\} \subset V_{d_1} \subset \cdots \subset V_{d_r} \subset V\}.$$

Flag Variety

There is a sequence of *tautological vector bundles* on the flag variety:

$$0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_{n-1} \subset \mathcal{V}_n = \mathbb{C}^n \times \mathrm{F}\ell(n)$$

where the fiber of $\mathcal{V}_i$ above a flag $(V_1 \subset \cdots \subset V_{n-1})$ is V_i. This gives the line bundles $\mathcal{L}_i = \mathcal{V}_i/\mathcal{V}_{i-1}$.

Recall that the *first Chern class* $c_1(\mathcal{L}) \in H^2(\mathrm{F}\ell(n))$ of a line bundle $\mathcal{L}$ is the cohomology class corresponding to the divisor of zeros and poles of $\mathcal{L}$. We will go back to these classes later on.

Schubert Cells

Fix a flag $\mathcal{F} = (F_1 \subset \cdots \subset F_n = V)$ with $F_i = \langle e_1, \ldots, e_i \rangle$. Given a basis $v_1, \ldots, v_n$ corresponding to the flag ($V_i = \langle v_1, \ldots, v_i \rangle$), we can express it as the rows of a $n \times n$ matrix. We then multiply by a lower-triangular matrix to put it in *row-echelon form*.

Example

$$\begin{pmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

A matrix is in row-echelon form if every column/row has exactly one leading 1's, and there are 0's below and to the right of each 1's.

Schubert Cells

We embed S_n into $\mathrm{GL}(n)$ by sending a permutation w to the matrix where the $(i, w(i))$ entries has 1's and 0's otherwise. Our goal is to define a *Schubert cell* $\Omega_w^\circ(\mathcal{F})$ consisting of matrices whose row-echelon form has the same position of leading 1's.

Definition

The Schubert cell $\Omega_w^\circ(\mathcal{F})$ is the collection of flags $V_\bullet \in \mathrm{F}\ell(n)$ that satisfy

$$\dim(V_p \cap F_q) = \#\{i \leqslant p : w(i) \leqslant q\} \text{ for all } 1 \leqslant p, q \leqslant n.$$

Example

$$\Omega_{4132}^\circ = \left\{ \begin{pmatrix} * & * & * & 1 \\ 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\} \simeq \mathbb{C}^4.$$

Schubert Cells

Each Schubert cell $\Omega_w^\circ(\mathcal{F})$ can also be described as the B-orbit of w. In other words, we have

$$\Omega_w^\circ = BwB/B.$$

The cell $\Omega_w^\circ(\mathcal{F})$ is isomorphic to $\mathbb{C}^{\ell(w)}$, where

$$\ell(w) = \#\{i < j : w(i) > w(j)\}$$

is the *length* of w.

We define the *Schubert variety* $\Omega_w(\mathcal{F})$ to be the closure of $\Omega_w(\mathcal{F})$ in the Zariski topology of $\mathrm{F}\ell(n)$. It consists of flags $V_\bullet$ satisfying

$$\dim(V_p \cap F_q) \geqslant r_w(p, q) \quad \text{for all } 1 \leqslant p, q \leqslant n$$

where $r_w(p, q) = \#\{i \leqslant p : w(i) \leqslant q\}$.

Bruhat Order

Question

Is the Schubert variety $\Omega_w(\mathcal{F})$ a disjoint union of Schubert cells?

Yes!

To state the containment condition, we need to introduce a partial order on S_n called the *Bruhat order*. For two permutations $u, v \in S_n$, define

$$u \leqslant v \qquad \text{if} \qquad r_u(p,q) \geqslant r_v(p,q) \text{ for all } 1 \leqslant p,q \leqslant n.$$

Recall that $r_w(p,q) = \#\{i \leqslant p : w(i) \leqslant q\}$.

Example

For S_3, $123 < 321$ since $r_{123} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} > r_{321} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.

Bruhat Order

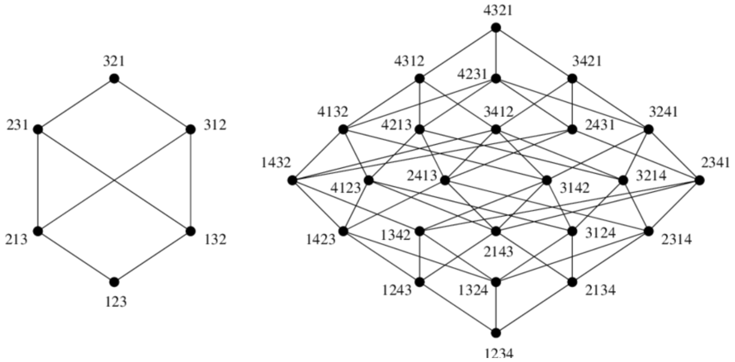

Figure: Bruhat order on S_3 and S_4

The Bruhat order has lots of interesting properties!

Schubert Stratification

Theorem

We have the following:

1. $\Omega_v \subset \Omega_w$ *if and only if* $v \leqslant w$.
2. $\Omega_w = \bigsqcup_{u \leqslant w} \Omega_w^\circ$.

Example

$$\Omega_{4312} = \overline{\left\{ \begin{pmatrix} * & * & * & 1 \\ * & * & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}} \supset \overline{\left\{ \begin{pmatrix} * & * & * & 1 \\ 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}} = \Omega_{4132}.$$

Schubert Stratification

When $w = w_0 = n(n-1)\dots 21$, we get the *Schubert stratification* of the flag variety

$$\mathrm{F}\ell(n) = \Omega_{w_0} = \bigsqcup_{w \in S_n} \Omega_w^\circ = \bigsqcup_{w \in S_n} BwB/B.$$

This also gives the *Bruhat decomposition* of the general linear group

$$\mathrm{GL}(n) = \bigsqcup_{w \in S_n} BwB.$$

Since the stratification consists of affine spaces, the cohomology $H^*(\mathrm{F}\ell(n))$ has a $\mathbb{Z}$-basis of *Schubert classes* $\sigma_w = [\Omega_w]$.

(just as with the Grassmannian, these Schubert classes don't depend on the choice of flag)

Presentation of $H^*(\mathrm{F}\ell(n))$

Question

Can we describe $H^*(\mathrm{F}\ell(n))$ as a (quotient of) a polynomial ring?

Recall the line bundles $\mathcal{L}_i = \mathcal{V}_i/\mathcal{V}_{i-1}$ that are quotients of tautological bundles. Let $x_i = -c_1(\mathcal{L}_i) \in H^2(\mathrm{F}\ell(n))$.

Theorem

The cohomology ring $H^*(\mathrm{F}\ell(n))$ is generated by the classes $x_1, \dots, x_n$ satisfying the relations $e_k(x_1, \dots, x_n) = 0$ for all $1 \leqslant k \leqslant n$. In other words, we have

$$H^*(\mathrm{F}\ell(n)) = \mathbb{Z}[x_1, \dots, x_n]/(e_1(x), \dots, e_n(x)).$$

Furthermore, the classes $x_1^{i_1} \dots x_n^{i_n}$ with exponents $i_j \leqslant n - j$ form a $\mathbb{Z}$-basis of $H^*(\mathrm{F}\ell(n))$.

Presentation of $H^*(\mathrm{F}\ell(n))$

Proof sketch

The generation statement comes from the fact that we can write $\mathrm{F}\ell(n)$ as a sequence of projective bundles

$$\mathrm{F}\ell(n) = \mathbb{P}(V/F_{n-1}) \to \mathbb{P}(V/F_{n-2}) \to \cdots \to \mathbb{P}(V/F_1) \to \mathbb{P}(V).$$

The relations $e_k(x_1, \ldots, x_n) = 0$ hold because $x_1, \ldots, x_n$ are the "Chern roots" of the trivial bundle $\mathcal{V} = V \times \mathrm{F}\ell(n)$. This means that

$$0 = c(\mathcal{V}) = \prod_{i=1}^{n} c(\mathcal{L}_i) = \prod_{i=1}^{n} (1 - x_i).$$

Schubert Polynomials

From the theorem, there exists polynomials $\mathfrak{S}_w \in \mathbb{Z}[X_1, \dots, X_n]$, homogeneous of degree $\ell(w)$, for each $w \in S_n$ such that

$$\mathfrak{S}_w(x_1, \dots, x_n) = \sigma_w.$$

These are the *Schubert polynomials*.

Example

For a simple transposition $s_i = (i, i+1)$ we can compute

$$\sigma_{s_i} = x_1 + \cdots + x_i,$$

or that $x_i = \sigma_{s_i} - \sigma_{s_{i-1}}$ for all i.

The Schubert polynomials can be computed by using an difference operator on the space of polynomials, as we now explain.

Divided Difference Operators

Recall that S_n acts on $\mathbb{Z}[X_1, \ldots, X_n]$ by permuting variables. For $1 \leqslant i \leqslant n-1$, define the *divided difference operator* ∂_i on $\mathbb{Z}[X_1, \ldots, X_n]$ by

$$\partial_i(P) = \frac{P - s_i(P)}{X_i - X_{i+1}}.$$

Example

$$\partial_i(X_i^a X_{i+1}^b) = \begin{cases} X_i^{a-1} X_{i+1}^b + \cdots + X_i^b X_{i+1}^{a-1} & \text{if } a > b \\ 0 & \text{if } a = b \\ -X_i^a X_{i+1}^{b-1} - \cdots - X_i^{b-1} X_{i+1}^a & \text{if } a < b \end{cases}.$$

Divided Difference Operators

Theorem

For any $1 \leqslant i \leqslant n-1$, we have

$$\partial_i(\mathfrak{S}_w) = \begin{cases} \mathfrak{S}_{ws_i} & \text{if } w(i) > w(i+1) \\ 0 & \text{if } w(i) < w(i+1) \end{cases}.$$

From the above, we can deduce all Schubert polynomials from the trivial one $\mathfrak{S}_{\text{id}} = 1$. For instance, $\mathfrak{S}_{s_i} = X_1 + \cdots + X_i$ since $\partial_j(\mathfrak{S}_{s_i}) = 1$ if $j = i$ and 0 otherwise.

In most cases, it is better to calculate from top down, i.e. apply ∂_i's to the top Schubert polynomial

$$\mathfrak{S}_{w_0} = X_1^{n-1} X_2^{n-2} \dots X_{n-2}^2 X_{n-1}.$$

Divided Difference Operators

Example

We calculate $\mathfrak{S}_{41352}$. There are several ways of getting there from $\mathfrak{S}_{54321}$; we choose $\partial_3 \circ \partial_2 \circ \partial_3 \circ \partial_1 \circ \partial_4$. This gives

$$\begin{aligned}
\mathfrak{S}_{54321} = X_1^4X_2^3X_3^2X_4 &\xrightarrow{\partial_4} \mathfrak{S}_{54312} = X_1^4X_2^3X_3^2 \\
&\xrightarrow{\partial_1} \mathfrak{S}_{45312} = X^{13}X_2^3X_3^2 \\
&\xrightarrow{\partial_3} \mathfrak{S}_{45132} = X_1^3X_2^3X_3 + X_1^3X_2^3X_4 \\
&\xrightarrow{\partial_2} \mathfrak{S}_{41532} = X_1^3X_2^2X_3 + X_1^3X_2X_3^2 + X_1^3X_2^2X_4 \\
&\qquad\qquad + X_1^3X_2X_3X_4 + X_1^3X_3^3X_4 \\
&\xrightarrow{\partial_3} \mathfrak{S}_{41532} = X_1^3X_2X_3 + X_1^3X_2X_4 + X_1^3X_3X_4.
\end{aligned}$$

Projection to Grassmannians

We have the projection $p : \mathrm{F}\ell(n) \to \mathrm{Gr}(k, n)$ sending $(V_1 \subset \cdots \subset V_{n-1}) \mapsto V_k$. This gives a pullback map

$$p^* : H^*(\mathrm{Gr}(k, n) \to H^*(\mathrm{F}\ell(n)).$$

What are the image of the Schubert classes σ_λ's of the Grassmannian?

Definition

A permutation $w \in S_n$ is called *Grassmannian* if it has only one descent. In other words, there exists some k such that $w(i) < w(i+1)$ for all $i \neq k$. For such a permutation, define

$$\lambda = (w(k) - k, w(r-1) - (r-1), \ldots, w(2) - 2, w(1) - 1).$$

Projection to Grassmannians

Let $p : \mathrm{F}\ell(n) \to \mathrm{Gr}(k, n)$ be the projection.

Theorem

If w is a Grassmannian permutation with unique descent k, then

$$\mathfrak{S}_w = s_\lambda(X_1, \ldots, X_k).$$

Furthermore, we have $p^{-1}(\Omega_\lambda) = \Omega_w$ and $p^*(\sigma_\lambda) = \sigma_w$.

Schubert Structure Constants

Similar to the cohomology structure of $H^*(\mathrm{Gr}(k,n))$, we have the following properties for $H^*(\mathrm{F}\ell(n))$.

- For $v, w \in S_n$ such that $\ell(v) + \ell(w) = \dim(\mathrm{F}\ell(n))$:

$$\mathfrak{S}_w \cdot \mathfrak{S}_v = \begin{cases} 1 & \text{if } v = w_0 w, \\ 0 & \text{otherwise.} \end{cases}$$

- Monk's formula:

$$\mathfrak{S}_{s_i} \cdot \mathfrak{S}_w = \sum_v \mathfrak{S}_v,$$

  where the sum is over all v obtained by w by interchanging a pair (p,q) with $1 \leqslant p \leqslant r < q \leqslant m$ such that $\ell(w \cdot (p,q)) = \ell(w) + 1$.

Schubert Structure Constants

However, there are no positive combinatorial description of general Schubert structure constants c_{vw}^{u} satisfying

$$\mathfrak{S}_v\mathfrak{S}_w = \sum_u c_{vw}^{u}\mathfrak{S}_u.$$

There is actually a formula by AJS, but it's not actually positive!

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