Schubert Calculus Day 3: The Littlewood-Richardson Rule

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Review

Last time, we investigated the structure of the cohomology ring $H^*(\mathrm{Gr}(k,n))$, which has a $\mathbb{Z}$-basis of Schubert classes $\sigma_\lambda = [\Omega_\lambda]$ for partitions $\lambda \subset (k^{n-k})$. We proved two formulas:

- For $\lambda, \mu \subset (k^{n-k})$ such that $|\lambda| + |\mu| = k(n-k)$,

$$\sigma_\lambda \cdot \sigma_\mu = \begin{cases} 1 & \text{if } \lambda_i + \mu_{k+1-i} = n-k \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

- For $1 \leqslant b \leqslant k$ and $\lambda \subset (k^{n-k})$,

$$\sigma_b \cdot \sigma_\lambda = \sum_{\mu = \lambda + (\text{horizontal strip of length } b)} \sigma_\mu$$

where $\sigma_\mu = 0$ if $\mu \not\subset (k^{n-k})$.

Review

Today, we will determine the multiplication rule in general. That is, given $\lambda, \mu \subset (k^{n-k})$, how do we determine the structure coefficients $c_{\lambda\mu}^{\nu}$ such that

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{|\nu|=|\lambda|+|\mu|} c_{\lambda\mu}^{\nu} \sigma_\nu \quad ?$$

These coefficients $c_{\lambda\mu}^{\nu}$ enumerate certain combinatorial objects as specified by the *Littlewood-Richardson rule*. To state this rule, we need to familiarize ourselves with the language of Young tableaux.

Young Tableaux

Definition

Given a partition λ, a *semi-standard Young tableau* of shape λ is a filling of the boxes in the Young diagram of λ with positive integers such that

- the entries in each row is non-decreasing,
- the entries in each column is (strictly) increasing.

Denote by $\mathsf{SSYT}(\lambda, n)$ the set of semi-standard Young tableau of shape λ whose entries do not exceed n. If the entries in a tableau T form a permutation of $\{1, 2, \ldots, |\lambda|\}$, we call T *standard*.

Example

$$\mathsf{SSYT}((3,1),2) = \left\{ \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \right\}.$$

Young Tableaux

For any n, let

$$R_{[n]} = \bigoplus_{\lambda} \bigoplus_{T \in \mathrm{SSYT}(\lambda, n)} \mathbb{Z}T.$$

Note that the first sum is finite since $\mathrm{SSYT}(\lambda, n) \neq \emptyset$ only when λ has at most n rows.

We will define a multiplication operator on $R_{[n]}$, making it a (non-commutative) ring called the *tableau ring*. This operator can be defined in multiple equivalent ways, as we shall show.

Row Insertion

Given a tableau T and a positive integer x, we can insert x into T to form a new tableau $T' = T \leftarrow x$ as follows.

Row Insertion Algorithm

Let $y = x$. Repeat for every row of T starting from the top:

1. If y is at least as large as all entries in the row, add y to the end of the row. Stop the algorithm.
2. If not, find the left-most entry z in the row larger than y. Put y in this box and update $y = z$.

Example

For a single row, we have

$$\boxed{1}\boxed{2}\boxed{3}\boxed{3} \quad \leftarrow \quad 2 = \boxed{1}\boxed{2}\boxed{2}\boxed{3} \qquad 3$$

Row Insertion

Example

$$\begin{ytableau} 1 & 2 & 3 & 3 \\ 2 & 3 & 5 & 5 \\ 4 & 5 & 6 \end{ytableau} \xleftarrow{\text{1st row}} 2 \quad = \begin{ytableau} 1 & 2 & 2 & 3 \\ 2 & 3 & 5 & 5 \\ 4 & 5 & 6 \end{ytableau} \xleftarrow{\text{2nd row}} 3$$

$$= \begin{ytableau} 1 & 2 & 2 & 3 \\ 2 & 3 & 3 & 5 \\ 4 & 5 & 6 \end{ytableau} \xleftarrow{\text{3rd row}} 5 \quad = \begin{ytableau} 1 & 2 & 2 & 3 \\ 2 & 3 & 3 & 5 \\ 4 & 5 & 5 \end{ytableau} \xleftarrow{\text{4th row}} 6$$

$$= \begin{ytableau} 1 & 2 & 2 & 3 \\ 2 & 3 & 3 & 5 \\ 4 & 5 & 5 \\ 6 \end{ytableau}$$

Row Insertion

For a word $w = w_1 \dots w_n$ with each entry a positive integer, define $T \leftarrow w$ to be

$$((((T \leftarrow w_1) \leftarrow w_2) \dots) \leftarrow w_n).$$

Given a tableau T, define its *reading word* $w(T)$ to be the word formed by concatenating the rows of T in reverse order. For example,

$$w\left(\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 3 \\ \hline 2 & 3 & 5 & 5 \\ \hline 4 & 5 & 6 & \multicolumn{1}{c}{} \\ \cline{1-3}\end{array}\right) = 45623551233.$$

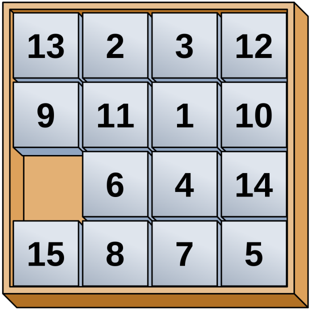
We now define the product of two tableaux T, U by

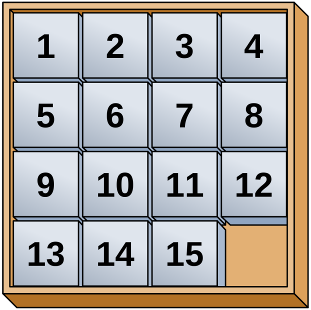
$$T \cdot^{\text{row insert}} U = T \leftarrow w(U).$$

(note that it's unclear whether this operation is associative)

Jeu-de-taquin

Jeu-de-taquin (literally "teasing game") is the French name for the fifteen puzzle.





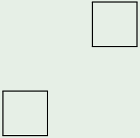
Jeu-de-taquin

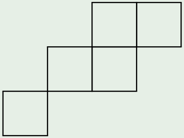
Definition

Given two partitions $\lambda \supset \mu$, define the *skew shape* λ/μ to be the diagram obtained by removing μ from the Young diagram of λ.

An *inner corner* of λ/μ is a box in the (deleted) μ such that the boxes below and to its right are not in μ. An *outer corner* of λ/μ is a box in λ such that the boxes below and to its right are not in λ.

Example

$(3,2,1)/(2,2) =$ 

$(4,3,1)/(2,1) =$ 

Jeu-de-taquin

Given a skew shape λ/μ, we can define a *skew semi-standard Young tableau* T of shape λ/μ to be a filling of λ/μ with positive integers satisfying the semi-standard conditions. Denote by $SSYT(\lambda/\mu, n)$ the set of these tableaux with entries at most n.

Our goal is now to "slide" the entries of T into the empty space μ such that the result is some SSYT T' of some shape ν. Each slide must be according to the following rule:

$$\begin{array}{cc} & \boxed{a} \\ \boxed{b} & \end{array} \quad \Rightarrow \quad \boxed{b}\boxed{a} \quad \text{if } a \geqslant b, \qquad \begin{array}{cc} & \boxed{a} \\ \boxed{b} & \end{array} \quad \Rightarrow \quad \begin{array}{c} \boxed{a} \\ \boxed{b} \end{array} \quad \text{if } a < b.$$

Jeu-de-taquin

Definition

Let x be an inner corner of a skew shape λ/μ and let $T \in \mathrm{SSYT}(\lambda/\mu, n)$. A *jeu-de-taquin (jdt) forward slide* for x is a tableau T' of shape $\lambda - \{\text{some outer corner}\}/(\mu - \{x\})$ obtained as a result of repeatedly sliding x until it becomes an outer corner.

Example

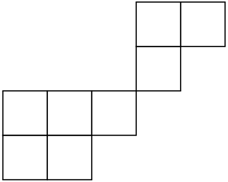
$$\begin{array}{cccc} & 2 & 3 & 4 \\ 1 & 3 & 4 & 6 \\ 3 & 5 & & \end{array} \Rightarrow \begin{array}{cccc} 1 & 2 & 3 & 4 \\ & 3 & 4 & 6 \\ 3 & 5 & & \end{array} \Rightarrow \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 3 & 3 & 4 & 6 \\ & 5 & & \end{array} \Rightarrow \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 3 & 3 & 4 & 6 \\ 5 & & & \end{array}$$

The *rectification* of a skew tableau $T \in \mathrm{SSYT}(\lambda/\mu, n)$ is the tableau $\mathrm{rect}(T)$ obtained after performing jdt slides on all inner corners of μ. This *doesn't depend* on the order of slides chosen.

Jeu-de-taquin

Given two shapes λ and μ, denote by $\lambda * \mu$ the skew shape formed by putting λ below and to the left of μ. In pictures, this is

$$(3,2) * (2,1) =$$



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Given tableaux T and U of shapes λ and μ respectively, let $T * U$ be the skew tableau of shape $\lambda * \mu$. We then define our second product on tableaux to be

$$T \cdot^{\text{jdt}} U = \text{rect}(T * U).$$

This product is easily seen to be associative.

Row Insertion vs. Jeu-de-taquin

Example

$$\begin{ytableau} 1 & 2 & 2 & 3 \\ 2 & 3 & 4 \end{ytableau} \cdot^{\text{jdt}} \begin{ytableau} 2 \end{ytableau} = \text{rect}\left(\begin{ytableau} \none & \none & \none & \none & 2 \\ 1 & 2 & 2 & 3 \\ 2 & 3 & 4 \end{ytableau} \right) = \begin{ytableau} 1 & 2 & 2 & 2 \\ 2 & 3 & 3 \\ 4 \end{ytableau}$$

The amazing thing is that these two product definitions are equivalent!

Theorem

For any $T, U \in R_{[n]}$, we have

$$T \cdot^{\text{row insert}} U = T \cdot^{\text{jdt}} U.$$

Symmetric Polynomials

Definition

A polynomial $P(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ is *symmetric* if for all $\sigma \in S_n$,

$$P(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = P(x_1, \ldots, x_n).$$

In other words, symmetric polynomials are elements of the invariant ring $\mathbb{Z}[x_1, \ldots, x_n]^{S_n} = \Lambda_n$.

Definition

Given a partition λ with at most n rows, define the *monomial symmetric polynomial* $m_\lambda(x)$ to be the sum of all monomials $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ over all *distinct* permutations $(\alpha_1, \ldots, \alpha_n)$ of $(\lambda_1, \ldots, \lambda_n)$.

Example

$m_{(3,1,1)}(x_1, x_2, x_3) = x_1^3 x_2 x_3 + x_1 x_2^3 x_3 + x_1 x_2 x_3^3$.

Symmetric Polynomials

From monomial symmetric polynomials, we can construct the following symmetric polynomials:

- The *k-th elementary symmetric polynomial*

$$e_k(x) = m_{(1^k)}(x) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} x_{i_1} \ldots x_{i_k},$$

- The *k-th power-sum symmetric polynomial*

$$p_k(x) = m_{(k)}(x) = x_1^k + \cdots + x_n^k,$$

- The *k-th complete homogeneous symmetric polynomial*

$$h_k(x) = \sum_{|\lambda|=k} m_\lambda(x) = \sum_{1 \leqslant i_1 \leqslant \ldots \leqslant i_k \leqslant n} x_{i_1} \ldots x_{i_k}.$$

Symmetric Polynomials

Theorem

Let $a \in \{e, p, h\}$. Then the ring of symmetric polynomials Λ_n is generated (as an algebra) by elements $a_k(x)$ for $1 \leqslant k \leqslant n$.

We now define *Schur polynomials*, which are a family of symmetric polynomials "interpolating" between $h_k(x)$ and $e_k(x)$.

Definition

Given a partition λ with at most n rows, the Schur polynomial $s_\lambda(x)$ is defined to be

$$s_\lambda(x) = \sum_{T \in \mathrm{SSYT}(\lambda, n)} x^T$$

where $x^T = x_1^{\#1\text{'s in } T} \dots x_n^{\#n\text{'s in } T}$.

Symmetric Polynomials

Example

$s_{(k)}(x) = h_k(x), \qquad s_{(1^k)}(x) = e_k(x).$
$s_{(2,2)}(x) = 2m_{(1,1,1,1)}(x) + m_{(2,1,1)}(x) + m_{(2,2)}(x) = e_2(x)^2 - e_1(x)e_3(x).$

The Schur polynomials satisfy many remarkable properties, including the following.

Theorem

1. *$s_\lambda(x_1, \ldots, x_n)$ is symmetric for all λ and n.*
2. *$s_\lambda(x)$ expands positively in the monomial basis $\{m_\lambda(x) \mid \lambda$ has at most n rows$\}$.*
3. *$\{s_\lambda(x_1, \ldots, x_n) \mid \lambda$ has at most n rows$\}$ form a $\mathbb{Z}$-basis for Λ_n.*

Schur polynomials are central objects in combinatorics and representation theory!

Symmetric Polynomials

We can define a similar family of objects in the tableau ring

$$S_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda, n)} T \in R_{[n]} \quad \text{for any } \lambda \text{ with at most } n \text{ rows.}$$

Let we can study the Schur polynomials through its tableau-theoretic shadow given by the ring homomorphism

$$R_{[n]} \to \mathbb{Z}[x_1, \ldots, x_n] \qquad \text{sending} \qquad T \mapsto x^T.$$

Since this map sends $S_\lambda \mapsto s_\lambda(x)$, any relation between S_λ's that holds in $R_{[n]}$ also holds between $s_\lambda(x)$'s in Λ_n.

Pieri's Rule for Tableaux

In particular, we can prove Pieri's rule in the tableau ring.

Theorem

For b a positive integer and λ a partition with at most n rows,

$$S_\lambda \cdot S_b = \sum_{\mu=\lambda+(\textit{horizontal strip of length } b)} S_\mu .$$

Proof sketch

The LHS consists of products $T \cdot B$ where $T \in \mathrm{SSYT}(\lambda, n)$ and $B \in \mathrm{SSYT}((b), n)$. Therefore, it suffices to show that for every $\mu = \lambda + (\text{horizontal strip of length } b)$ and every $U \in \mathrm{SSYT}(\mu, n)$, there is exactly one way for U to be written in the form $T \cdot B$ above. This could then be proved using properties of the row insertion algorithm.

Connection to Schubert classes

Since both Schur polynomials and Schubert classes satisfy Pieri's rule in their respective rings, we get a homomorphism

$$\Lambda_n \to H^*(\mathrm{Gr}(k,n)) \qquad \text{sending} \quad s_\lambda(x) \mapsto \sigma_\lambda,$$

where $\sigma_\lambda = 0$ if λ does not fit in the $k \times (n-k)$ rectangle.

Combining this with the map $\mathbb{Z}[S_\lambda] \to \Lambda_n$, we conclude that the structure constants of

$$S_\lambda S_\mu = \sum_\nu c^\nu_{\lambda\mu} S_\nu,$$

if they exist, are the same as the structure constants of multiplying Schubert classes.

Littlewood-Richardson Rule

To show that there exists constants $c_{\lambda\mu}^{\nu}$ satisfying

$$S_\lambda S_\mu = \sum_\nu c_{\lambda\mu}^{\nu} S_\nu,$$

it suffices to show that for every ν and every $V \in \mathrm{SSYT}(\nu, n)$, the number of ways that V can be written as $T \cdot U$ for $T \in \mathrm{SSYT}(\lambda, n)$ and $U \in \mathrm{SSYT}(\mu, n)$ is exactly $c_{\lambda\mu}^{\nu}$.

In other words, if we denote

$$\mathcal{T}(\lambda, \mu, V_0) = \{T \in \mathrm{SSYT}(\lambda, n), U \in \mathrm{SSYT}(\mu, n) \mid T \cdot U = V_0\}$$

then we need to show that $\mathcal{T}(\lambda, \mu, V_0) = \mathcal{T}(\lambda, \mu, V_1)$ for any $V_0, V_1 \in \mathrm{SSYT}(\nu, n)$. This is part of the Littlewood-Richardson rule.

Littlewood-Richardson Rule

In addition to the previous statement, the Littlewood-Richardson rule says that we can enumerate $\mathcal{T}(\lambda, \mu, V_0)$ in another way! Consider

$$\mathcal{S}(\nu, \lambda, U_0) = \{U \in \mathrm{SSYT}(\nu/\lambda, n) \mid \mathrm{rect}(U) = U_0\}.$$

Theorem (LR rule)

For any $V_0 \in \mathrm{SSYT}(\nu, n)$ and $U_0 \in \mathrm{SSYT}(\mu, n)$, there is a bijection

$$\mathcal{T}(\lambda, \mu, V_0) \leftrightarrow \mathcal{S}(\nu, \lambda, U_0).$$

Littlewood-Richardson rule

For any shape λ, there is a "minimal" tableau $T_{\min}(\lambda)$ where its i-th row has all its entries be i. Specializing the LR rule to these minimal tableaux gives us the following.

LR rule, second version

The constant $c^{\nu}_{\lambda\mu}$ is equal to both of the following:

1. The number of tableaux of shape $\lambda * \mu$ that rectifies to $T_{\min}(\nu)$.
2. The number of tableaux of shape ν/λ that rectifies to $T_{\min}(\mu)$.

Littlewood-Richardson rule

There is yet another form of the LR rule based on the reading word of a tableau. Call a word $w = w_1 \dots w_m$ *lattice* if for every $k \leqslant m$ and every $i \leqslant n$, $w_1 \dots w_k$ has at least as many i's as it has $(i+1)$'s. A word w is *reverse lattice* if its reverse is lattice.

Example

1123243 is lattice but 1123324 is not.

Definition

A tableau $T \in \mathrm{SSYT}(\lambda/\mu, n)$ is a *Littlewood-Richardson tableau* if its reading word is reverse lattice.

For example, the only LR tableau of straight shape λ is the minimal one. A key property we'll use is that if T is a LR tableau and T is jdt equivalent to T', then T' is also a LR tableau.

Littlewood-Richardson rule

LR rule, third version

The constant $c^{\nu}_{\lambda\mu}$ is equal to both of the following:

1. The number of Litlewood-Richardson tableaux of shape $\lambda * \mu$ and content ν.
2. The number of Littlewood-Richardson tableaux of shape ν/λ and content μ.

Example

There are 2 LR tableaux of shape $(5,4,3,2)/(3,3,1)$ and content $(3,2,1,1)$:

			1	1
			2	
	1	3		
2	4			

			1	1
			2	
	2	3		
1	4			

Littlewood-Richardson Rule

The LR rule was first proposed by Littlewood and Richardson in 1934; in their paper, they claimed a proof but it was incorrect. The first rigorous proofs were found by Schutzenberger (1977) and Thomas (1974), using the *Robinson-Schensted-Knuth (RSK) equivalence*.

The correspondence states that there is a bijection

$\{\text{two-row arrays of positive integers}\} \leftrightarrow \{\text{pairs of SSYTs of the same shape}\}.$

Following their proof, it also follows that the LR coefficients are commutative, i.e. $c^{\nu}_{\lambda\mu} = c^{\nu}_{\lambda\mu}$, which is not evident from our discussion.

Littlewood-Richardson Rule

The LR rule has been proved in many different ways, in addition to our proof sketch above. Among them are two worth mentioning for their novelty.

- Vakil (2005) proved the LR rule directly from the Schubert classes, using a degeneration argument.
- Knutson and Tao (2000) showed that the LR coefficient $c^{\nu}_{\lambda\mu}$ enumerates a class of objects called *puzzles* which are in bijection with LR tableaux. More on this on day 5!

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