Schubert Calculus Day 2: Schubert classes and Schur polynomials

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Table of Content

1. Review of Day 1
2. Ring Structure of $H^*(\text{Gr}(k,n))$
3. More Enumerative Geometry
Yesterday, we defined the Grassmannian $\text{Gr}(k, n)$ which parametrizes $k$-subspaces of $n$-dim space. It has a stratification into Schubert cells

$$\text{Gr}(k, n) = \bigsqcup_{\lambda \subset (k^{n-k})} \Omega^\circ_\lambda(\mathcal{F}),$$

where $\mathcal{F}$ is any complete flag. A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ determines a position $p_\lambda = (n - k + 1 - \lambda_1, n - k + 2 - \lambda_2, \ldots, n - \lambda_k)$ and

$$\Omega^\circ_\lambda(\mathcal{F}) = \{ U \in \text{Gr}(k, n) \mid \dim(U \cap F_j) = i \text{ for } p_i \leq j < p_{i+1} \}.$$

The cohomology ring $H^*(\text{Gr}(k, n))$ is generated in even degrees by the class $\sigma_{\lambda} = [\Omega_\lambda(\mathcal{F})]$ of the Schubert varieties. Here $\sigma_{\lambda} \in H^{2|\lambda|}(\text{Gr}(k, n))$ and is independent of the choice of flag.
Think of $\text{Gr}(2, 4)$ as parametrizing lines in $\mathbb{P}^3$. Fix a flag $p \subset \ell \subset H$ in $\mathbb{P}^3$.

### $\Omega_{(0,0)}$

$$\Omega_{(0,0)} = \{ \Lambda \mid \Lambda \cap H \neq \emptyset \}$$

$$= \left\{ \begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix} \right\}$$

### $\Omega_{(1,0)}$

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### $\Omega_{(2,0)}$

$$\Omega_{(2,0)} = \{ \Lambda \mid p \in \Lambda \}$$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{pmatrix} \right\}$$

### $\Omega_{(1,1)}$

$$\Omega_{(1,1)} = \{ \Lambda \mid \Lambda \subset H \}$$

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### $\Omega_{(2,1)}$

$$\Omega_{(2,1)} = \{ \Lambda \mid p \in \Lambda \subset H \}$$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{pmatrix} \right\}$$

### $\Omega_{(2,2)}$

$$\Omega_{(2,2)} = \{ \Lambda \mid \Lambda = \ell \}$$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}$$
Let $X$ be a nonsingular projective variety over $\mathbb{C}$, and let $V$, $W$ be two subvarieties of $X$. Let $Z_1, \ldots, Z_n$ be the irreducible components of $V \cap W$.

**Question**

What does the cup product $[V] \cdot [W]$ actually mean?

In “nice” situations, it will turn out to be exactly $[Z_1] + \cdots + [Z_n]$.

**Definition**

Let $X$, $V$, $W$, $Z_i$ be as above. We call the intersection $V \cap W$ proper if $\text{codim}(Z_i) = \text{codim}(V) + \text{codim}(W)$ for all $i$.

We call the intersection (generically) transverse if for each $i$, we have an open Zariski subset of $Z_i$ such that for all $z$ in that open subset,

$$T_z Z_i = T_z V \cap T_z W.$$
Theorem

If \( V \) intersects \( W \) transversely, then \([V] \cdot [W] = [Z_1] + \cdots + [Z_n]\).
If their intersection is only proper, then \([V] \cdot [W] = m_1[Z_1] + \cdots + m_n[Z_n]\)
for some multiplicities \( m_i \).

Thus in our Schubert calculus problems, it is essential that the intersection
of Schubert varieties are transverse.

Theorem (Kleiman’s transversality)

Suppose that an algebraic group \( G \) acts transitively on a variety \( X \) over \( \mathbb{C} \),
and \( A \subseteq X \) is a subvariety.

a) If \( B \subseteq X \) is another subvariety, then there is an open dense set of
\( g \in G \) such that \( gA \) is generically transverse to \( B \).
b) If \( G \) is affine, then \([gA] = [A]\) in \( H^*(X) \) for any \( g \in G \).
Intersection in Complementary Dimensions

Consider two partitions $\lambda, \mu \subset (k^{n-k})$ with $|\lambda| + |\mu| = k(n-k)$. We call them complementary if $\lambda_i + \mu_{k+1-i} = n-k$ for all $1 \leq i \leq k$. In other words, $\lambda$ and $\mu$ complement each other in the $k \times (n-k)$ rectangle.

**Theorem**

Let $\lambda$ and $\mu$ be as above, and $F, \tilde{F}$ be opposite flags. Then the Schubert varieties $\Omega_\lambda(F)$ and $\Omega_\mu(\tilde{F})$ intersect transversely at a unique point if $\lambda_i + \mu_{k+1-i} = n-k$ for all $i$, and are disjoint otherwise.

**Corollary**

We have

$$\sigma_\lambda \cdot \sigma_\mu = \begin{cases} 1 & \text{if } \lambda \text{ and } \mu \text{ are complementary}, \\ 0 & \text{otherwise}. \end{cases}$$
Intersection in Complementary Dimensions

Proof.

For $U \in \Omega_\lambda(F) \cap \Omega_\mu(\tilde{F})$, the dimension conditions are

\[
\begin{align*}
\dim(U \cap F_{n-k+i-\lambda_i}) &\geq i \text{ for all } i \\
\dim(U \cap \tilde{F}_{n-k+i-\mu_i}) &\geq i \text{ for all } i
\end{align*}
\]

We then combine the first condition for $i$ and the second condition for $k+1-i$ to get

\[
\dim(U \cap F_{n-k+i-\lambda_i}) + \dim(U \cap \tilde{F}_{n-k+(k+1-i)-\mu_{k+1-i}}) \geq k + 1.
\]

Since $\dim(U) = k$, we must have $\dim(F_{n-k+i-\lambda_i} \cap \tilde{F}_{n+1-i-\mu_{k+1-i}}) \geq 1$.

Because $F, \tilde{F}$ are transverse flags, this translates to $\lambda_i + \mu_{k+1-i} \leq n - k$ for all $i$. Since $|\lambda| + |\mu| < k(n - k)$, equality must happen for all $i$. \qed
What is the unique point of intersection?

\[
\begin{align*}
\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \ast & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \cap \left\{ \begin{pmatrix} 1 & \ast & 0 & \ast \\ 0 & 0 & 1 & \ast \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} &= \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.
\end{align*}
\]
Pieri’s Rule

Given a partition \( \lambda \), a \textit{horizontal strip} of length \( b \) for \( \lambda \) is a way to attach \( b \) more boxes to its Young diagram such that

- the result Young diagram is still a partition, and
- no two added boxes are in the same column.

If we call the resulting partition \( \mu \), then this is equivalent to saying that \( \mu \supset \lambda \), \( |\mu| = |\lambda| + b \) and \( \lambda_i \leq \mu_i \leq \lambda_{i-1} \) for all \( i \).

**Theorem (Pieri’s Rule)**

\[
\sigma_b \cdot \sigma_\lambda = \sum_{\mu = \lambda + (\text{horizontal strip of length } b)} \sigma_\mu.
\]
Pieri’s Rule

Proof

It suffices to show that for any partition $\mu$ with $|\mu| = |\lambda| + b$, we have

$$\sigma_b \sigma_\lambda \sigma_{\text{comp}(\mu)} = \begin{cases} 1 & \text{if } \mu = \lambda + \text{(horizontal strip of length } b) , \\ 0 & \text{otherwise.} \end{cases}$$

Here $\text{comp}(\mu)$ denotes the complementary partition to $\mu$. This further reduces to showing that for a pair of opposite flags $\mathcal{F}, \tilde{\mathcal{F}}$, and a general flag $\mathcal{V}$, $\Omega_\lambda(\mathcal{F}), \Omega_b(\mathcal{U})$ and $\Omega_{\text{comp}(\mu)}(\tilde{\mathcal{F}})$ intersect at exactly 1 point if $\lambda_i \leq \mu_i \leq \lambda_{i-1}$, and are disjoint otherwise.

Assume that their intersection is non-empty. Since $\Omega_\lambda(\mathcal{F}) \cap \Omega_{\text{comp}(\mu)}(\tilde{\mathcal{F}}) \neq \emptyset$, a similar reasoning to the previous Theorem shows that $\lambda_i + \text{comp}(\mu)_{k+1-i} \leq n - k$ for all $i$. This is equivalent to $\lambda_i \leq \mu_i$. 
Pieri’s Rule

Proof (continued)

The harder direction is to show that \( \mu_i \leq \lambda_{i-1} \). Recall that the Schubert conditions are

\[
\Omega_\lambda(F) = \{ \Lambda \mid \dim(\Lambda \cap F_{n-k+i-\lambda_i}) \geq i \text{ for all } i \},
\]

\[
\Omega_{\text{comp}(\mu)}(\bar{F}) = \{ \Lambda \mid \dim(\Lambda \cap \bar{F}_{i+\mu_{k+1-i}}) \geq i \text{ for all } i \}.
\]

Let \( C_i = F_{n-k+i-\lambda_i} \cap \bar{F}_{k+1-i+\mu_i} \), so either \( \dim(C_i) = \mu_i - \lambda_i + 1 \) for all \( i \). Then the dimension conditions imply that if \( \Lambda \in \Omega_\lambda(F) \cap \Omega_{\text{comp}(\mu)}(\bar{F}) \), then \( \Lambda \cap C_i \neq 0 \) for all \( i \). Let \( C = \text{span}(C_1, \ldots, C_k) \). A simple calculation shows that

\[
\dim(C) \leq \sum_{i=1}^{k} \dim(C_i) = \sum_{i=1}^{k} (c_i - a_i + 1) = k + b.
\]
We now use the description of $\Omega_b(U)$ as the set of $k$-planes meeting a general subspace $U = U_{n-k+1-b}$. If $\Lambda$ is in the triple intersection then we need $C \cap U \neq \emptyset$. Since $U$ is general, we need $\dim(C) \geq k + b$. Hence equality must occur, which implies that $C_1, \ldots, C_k$ are linearly independent. It is then a short check to show that this implies $\mu_i \leq \lambda_{i-1}$ for all $i$. 
Giambelli’s Formula

**Theorem**

For a partition \( \lambda = (\lambda_1, \ldots, \lambda_r) \subset (k^{n-k}) \), we have

\[
\sigma_\lambda = \det (\sigma_{\lambda_i+j-i})_{1 \leq i, j \leq r}.
\]

Here \( \sigma_a = 0 \) if \( a < 0 \).

**Corollary**

\( \text{H}^2(\text{Gr}(k, n)) \) is generated by the Schubert classes \( \sigma_1, \ldots, \sigma_k \).

This is a direct consequence of Pieri’s rule (plus induction and lots of cancellations).
Giambelli’s Formula

For $r = 2$ and $\lambda = (a, b)$:

$$\det \begin{pmatrix} \sigma_a & \sigma_{a+1} \\ \sigma_{b-1} & \sigma_b \end{pmatrix} = \sigma_a \sigma_b - \sigma_{b-1} \sigma_{a+1}$$

$$= (\sigma_{a+b} + \sigma_{a+b-1,1} + \cdots + \sigma_{a,b})$$

$$- (\sigma_{a+b} + \cdots + \sigma_{a+1,b-1})$$

$$= \sigma_{a,b}.$$

For $r = 3$ and $\lambda = (a, b, c)$:

$$\det \begin{pmatrix} \sigma_a & \sigma_{a+1} & \sigma_{a+2} \\ \sigma_{b-1} & \sigma_b & \sigma_{b+1} \\ \sigma_{c-2} & \sigma_{c-1} & \sigma_c \end{pmatrix} = \sigma_a \sigma_{b,c} - \sigma_{b-1} \sigma_{a+1,c} + \sigma_{c-2} \sigma_{a+1,b+1}$$

$$= \cdots$$

$$= \sigma_{a,b,c}.$$
Back to the Motivating Problem

**Question**

Given four lines $\ell_1, \ldots, \ell_4$ in $\mathbb{P}^3$ in general position, how many lines intersect all four?

Now that we know Pieri’s rule, we can compute

$$
\sigma_1^4 = (\sigma_{2,0} + \sigma_{1,1})^2 = \sigma_{2,2} + \sigma_{2,2} = 2\sigma_{2,2}.
$$

Hence the answer is 2. We will sketch another proof that does not use Schubert calculus.

**Lemma**

Given a point $p$ and two lines $\ell_1, \ell_2$ in $\mathbb{P}^3$ in general position. Then there exists a unique line through $p$ that intersect both $\ell_1$ and $\ell_2$.

For the proof, pick a general plane $H$ and projects $\ell_1, \ell_2$ onto it from $p$. Connect $p$ with the point of intersection on $H$ to get the desired line.
For each point $p \in \ell_3$, let $M_p$ be the unique line passing through $p$ and intersect both $\ell_1$ and $\ell_2$. Let $Q = \bigcup_p M_p$.

**Key Fact**

The lines $M_p$ are disjoint, and $Q$ is a quadric surface.

Thus, the fourth general line $\ell_4$ will intersect $Q$ at two points, corresponding to two lines $M_p, M_q$. 

**Figure**: Figure 3.8 in “3264 and all that”
A Generalization

Question

Given four smooth curves $C_1, \ldots, C_4$ in $\mathbb{P}^3$ of degree $d_1, \ldots, d_4$ respectively and in general position, how many lines intersect all four curves?

The proof is similar to the previous problem. For a curve $C$ in $\mathbb{P}^3$, let

$$\Gamma_C = \{ \ell \in \text{Gr}(2, 4) \mid \ell \cap C \neq \emptyset \}.$$

We can show that this is a subvariety of $\text{Gr}(2, 4)$ of codimension 1, so $[\Gamma_C] = d\sigma_1$ for some integer $d$. To determine $d$, we use the method of underdetermined coefficients. In other words, we multiply

$$[\Gamma_C]\sigma_{2,1} = d\sigma_1\sigma_{2,1} = d\sigma_{2,2}.$$

Hence $d$ is the number of points in the intersection $\Gamma_C \cap \Omega_{2,1}(\mathcal{F})$, assuming it is transverse. This can be computed to be $\deg(C)$.
A Generalization

Question

Given four smooth curves $C_1, \ldots, C_4$ in $\mathbb{P}^3$ of degree $d_1, \ldots, d_4$ respectively and in general position, how many lines intersect all four curves?

Hence the intersection number is

$$\prod_i [\Gamma_{C_i}] = 2d_1 d_2 d_3 d_4.$$
For general partitions $\lambda, \mu \subset (k^{n-k})$, the product $\sigma_\lambda \cdot \sigma_\mu$ is a linear combination of $\{\sigma_\nu \mid |\nu| = |\lambda| + |\mu|\}$. In other words,

$$\sigma_\lambda \sigma_\mu = \sum_{|\nu| = |\lambda| + |\nu|} c^\nu_{\lambda\mu} \sigma_\nu$$

for some integers $c^\nu_{\lambda\mu}$.

**Question**

Is there an algorithm/combinatorial formula to compute these coefficient?

The answer to this question is the *Littlewood-Richardson rule*, and requires us to take a detour into the world of symmetric polynomials and Young tableaux.
References

- **Sara Billey.** “Tutorial on Schubert Varieties and Schubert Calculus”. In: (2013).
- **David Eisenbud and Joe Harris.** *3264 and all that A second course in algebraic geometry*. Cambridge University Press, 2016.