Schubert Calculus Day 2: Schubert classes and Schur polynomials

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May 2021

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Review

Yesterday, we defined the Grassmannian $\mathrm{Gr}(k,n)$ which parametrizes k-subspaces of n-dim space. It has a stratification into Schubert cells

$$\mathrm{Gr}(k,n) = \bigsqcup_{\lambda \subset (k^{n-k})} \Omega^{\circ}_{\lambda}(\mathcal{F}),$$

where $\mathcal{F}$ is any complete flag. A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ determines a position $p_\lambda = (n-k+1-\lambda_1, n-k+2-\lambda_2, \ldots, n-\lambda_k)$ and

$$\Omega^{\circ}_{\lambda}(\mathcal{F}) = \{U \in \mathrm{Gr}(k,n) \mid \dim(U \cap F_j) = i \text{ for } p_i \leqslant j < p_{i+1}\}.$$

The cohomology ring $H^*(\mathrm{Gr}(k,n))$ is generated in even degrees by the class $\sigma_\lambda = [\Omega_\lambda(\mathcal{F})]$ of the Schubert varieties. Here $\sigma_\lambda \in H^{2|\lambda|}(\mathrm{Gr}(k,n))$ and is independent of the choice of flag.

Schubert Varieties for Gr(2, 4)

Think of Gr(2, 4) as parametrizing lines in $\mathbb{P}^3$. Fix a flag $p \subset \ell \subset H$ in $\mathbb{P}^3$.

$$\Omega_{(0,0)} = \overline{\{\Lambda \mid \Lambda \cap H \neq \emptyset\}} = \left\{\begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix}\right\}$$

$$\Omega_{(1,0)} = \overline{\{\Lambda \mid \Lambda \cap \ell \neq \emptyset\}} = \left\{\begin{pmatrix} * & 1 & \boxed{0} & 0 \\ * & 0 & * & 1 \end{pmatrix}\right\}$$

$$\Omega_{(2,0)} = \overline{\{\Lambda \mid p \in \Lambda\}} = \left\{\begin{pmatrix} 1 & \boxed{0} & \boxed{0} & 0 \\ 0 & * & * & 1 \end{pmatrix}\right\}$$

$$\Omega_{(1,1)} = \overline{\{\Lambda \mid \Lambda \subset H\}} = \left\{\begin{pmatrix} * & 1 & 0 & \boxed{0} \\ * & 0 & 1 & \boxed{0} \end{pmatrix}\right\}$$

$$\Omega_{(2,1)} = \overline{\{\Lambda \mid p \in \Lambda \subset H\}} = \left\{\begin{pmatrix} 1 & \boxed{0} & 0 & \boxed{0} \\ 0 & * & 1 & \boxed{0} \end{pmatrix}\right\}$$

$$\Omega_{(2,2)} = \overline{\{\Lambda \mid \Lambda = \ell\}} = \left\{\begin{pmatrix} 1 & 0 & \boxed{0} & \boxed{0} \\ 0 & 1 & \boxed{0} & \boxed{0} \end{pmatrix}\right\}$$

From Cohomology to Point Counting

Let X be a nonsingular projective variety over $\mathbb{C}$, and let V, W be two subvarieties of X. Let $Z_1, \dots, Z_n$ be the irreducible components of $V \cap W$.

Question

What does the cup product $[V] \cdot [W]$ actually mean?

In "nice" situations, it will turn out to be exactly $[Z_1] + \cdots + [Z_n]$.

Definition

Let X, V, W, Z_i be as above. We call the intersection $V \cap W$ *proper* if $\operatorname{codim}(Z_i) = \operatorname{codim}(V) + \operatorname{codim}(W)$ for all i.
We call the intersection *(generically) transverse* if for each i, we have an open Zariski subset of Z_i such that for all z in that open subset,

$$T_z Z_i = T_z V \cap T_z W.$$

From Cohomology to Point Counting

Theorem

If V intersects W transversely, then $[V] \cdot [W] = [Z_1] + \cdots + [Z_n]$. If their intersection is only proper, then $[V] \cdot [W] = m_1[Z_1] + \cdots + m_n[Z_n]$ for some multiplicities m_i.

Thus in our Schubert calculus problems, it is essential that the intersection of Schubert varieties are transverse.

Theorem (Kleiman's transversality)

Suppose that an algebraic group G acts transitively on a variety X over $\mathbb{C}$, and $A \subset X$ is a subvariety.

- a) *If $B \subset X$ is another subvariety, then there is an open dense set of $g \in G$ such that gA is generically transverse to B.*
- b) *If G is affine, then $[gA] = [A]$ in $H^*(X)$ for any $g \in G$.*

Intersection in Complementary Dimensions

Consider two partitions $\lambda, \mu \subset (k^{n-k})$ with $|\lambda| + |\mu| = k(n-k)$. We call them *complementary* if $\lambda_i + \mu_{k+1-i} = n-k$ for all $1 \leqslant i \leqslant k$. In other words, λ and μ complement each other in the $k \times (n-k)$ rectangle.

Theorem

Let λ and μ be as above, and $\mathcal{F}$, $\widetilde{\mathcal{F}}$ be opposite flags. Then the Schubert varieties $\Omega_\lambda(\mathcal{F})$ and $\Omega_\mu(\widetilde{\mathcal{F}})$ intersect transversely at a unique point if $\lambda_i + \mu_{k+1-i} = n-k$ for all i, and are disjoint otherwise.

Corollary

We have

$$\sigma_\lambda \cdot \sigma_\mu = \begin{cases} 1 & \text{if } \lambda \text{ and } \mu \text{ are complementary,} \\ 0 & \text{otherwise.} \end{cases}$$

Intersection in Complementary Dimensions

Proof.

For $U \in \Omega_\lambda(\mathcal{F}) \cap \Omega_\mu(\widetilde{\mathcal{F}})$, the dimension conditions are

$$\begin{cases} \dim(U \cap F_{n-k+i-\lambda_i}) \geqslant i \text{ for all } i \\ \dim(U \cap \widetilde{F}_{n-k+i-\mu_i}) \geqslant i \text{ for all } i \end{cases}.$$

We then combine the first condition for i and the second condition for $k+1-i$ to get

$$\dim(U \cap F_{n-k+i-\lambda_i}) + \dim(U \cap \widetilde{F}_{n-k+(k+1-i)-\mu_{k+1-i}}) \geqslant k+1.$$

Since $\dim(U) = k$, we must have $\dim(F_{n-k+i-\lambda_i} \cap \widetilde{F}_{n+1-i-\mu_{k+1-i}}) \geqslant 1$. Because $\mathcal{F}, \widetilde{\mathcal{F}}$ are transverse flags, this translates to $\lambda_i + \mu_{k+1-i} \leqslant n-k$ for all i. Since $|\lambda| + |\mu| < k(n-k)$, equality must happen for all i. □

Intersection in Complementary Dimensions

What is the unique point of intersection?

$$\overline{\left\{\begin{pmatrix}1 & \boxed{0} & 0 & \boxed{0} \\ 0 & * & 1 & \boxed{0}\end{pmatrix}\right\}} \cap \overline{\left\{\begin{pmatrix}1 & * & 0 & * \\ 0 & \boxed{0} & 1 & *\end{pmatrix}\right\}} = \left\{\begin{pmatrix}1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0\end{pmatrix}\right\}.$$

Pieri's Rule

Given a partition λ, a *horizontal strip* of length b for λ is a way to attach b more boxes to its Young diagram such that

- the result Young diagram is still a partition, and
- no two added boxes are in the same column.

If we call the resulting partition μ, then this is equivalent to saying that $\mu \supset \lambda$, $|\mu| = |\lambda| + b$ and $\lambda_i \leqslant \mu_i \leqslant \lambda_{i-1}$ for all i.

Theorem (Pieri's Rule)

For any $b \leqslant k$ and any partition $\lambda \subset (k^{n-k})$, we have

$$\sigma_b \cdot \sigma_\lambda = \sum_{\mu=\lambda+(\text{horizontal strip of length } b)} \sigma_\mu .$$

Pieri's Rule

Proof

It suffices to show that for any partition μ with $|\mu| = |\lambda| + b$, we have

$$\sigma_b \sigma_\lambda \sigma_{\mathrm{comp}(\mu)} = \begin{cases} 1 & \text{if } \mu = \lambda + (\text{horizontal strip of length } b), \\ 0 & \text{otherwise.} \end{cases}$$

Here comp(μ) denotes the complementary partition to μ.This further reduces to showing that for a pair of opposite flags $\mathcal{F}$, $\widetilde{\mathcal{F}}$, and a general flag $\mathcal{V}$, $\Omega_\lambda(\mathcal{F}), \Omega_b(\mathcal{U})$ and $\Omega_{\mathrm{comp}(\mu)}(\widetilde{\mathcal{F}})$ intersect at exactly 1 point if $\lambda_i \leqslant \mu_i \leqslant \lambda_{i-1}$, and are disjoint otherwise.

Assume that their intersection is non-empty. Since $\Omega_\lambda(\mathcal{F}) \cap \Omega_{\mathrm{comp}(\mu)}(\widetilde{\mathcal{F}}) \neq \emptyset$, a similar reasoning to the previous Theorem shows that $\lambda_i + \mathrm{comp}(\mu)_{k+1-i} \leqslant n - k$ for all i. This is equivalent to $\lambda_i \leqslant \mu_i$.

Pieri's Rule

Proof (continued)

The harder direction is to show that $\mu_i \leqslant \lambda_{i-1}$. Recall that the Schubert conditions are

$$\Omega_\lambda(\mathcal{F}) = \{\Lambda \mid \dim(\Lambda \cap F_{n-k+i-\lambda_i}) \geqslant i \text{ for all } i\},$$

$$\Omega_{\mathrm{comp}(\mu)}(\widetilde{F}) = \{\Lambda \mid \dim(\Lambda \cap \widetilde{F}_{i+\mu_{k+1-i}}) \geqslant i \text{ for all } i.$$

Let $C_i = F_{n-k+i-\lambda_i} \cap \widetilde{F}_{k+1-i+\mu_i}$, so either $\dim(C_i) = \mu_i - \lambda_i + 1$ for all i. Then the dimension conditions imply that if $\Lambda \in \Omega_\lambda(\mathcal{F}) \cap \Omega_{\mathrm{comp}(\mu)}(\widetilde{F})$, then $\Lambda \cap C_i \neq 0$ for all i. Let $C = \mathrm{span}(C_1, \ldots, C_k)$. A simple calculation shows that

$$\dim(C) \leqslant \sum_{i=1}^{k} \dim(C_i) = \sum_{i=1}^{k} (c_i - a_i + 1) = k + b.$$

Pieri's Rule

Proof (continued)

We now use the description of $\Omega_b(\mathcal{U})$ as the set of k-planes meeting a general subspace $U = U_{n-k+1-b}$. If Λ is in the triple intersection then we need $C \cap U \neq \emptyset$. Since U is general, we need $\dim(C) \geqslant k + b$. Hence equality must occur, which implies that $C_1, \ldots, C_k$ are linearly independent. It is then a short check to show that this implies $\mu_i \leqslant \lambda_{i-1}$ for all i.

Giambelli's Formula

Theorem

For a partition $\lambda = (\lambda_1, \ldots, \lambda_r) \subset (k^{n-k})$, *we have*

$$\sigma_\lambda = \det\left(\sigma_{\lambda_i + j - i}\right)_{1 \leqslant i,j \leqslant r}.$$

Here $\sigma_a = 0$ *if* $a < 0$.

Corollary

$H^{2*}(Gr(k, n))$ *is generated by the Schubert classes* $\sigma_1, \ldots, \sigma_k$.

This is a direct consequence of Pieri's rule (plus induction and lots of cancellations).

Giambelli's Formula

For $r = 2$ and $\lambda = (a, b)$:

$$\det \begin{pmatrix} \sigma_a & \sigma_{a+1} \\ \sigma_{b-1} & \sigma_b \end{pmatrix} = \sigma_a \sigma_b - \sigma_{b-1}\sigma_{a+1}$$

$$= (\sigma_{a+b} + \sigma_{a+b-1,1} + \cdots + \sigma_{a,b})$$

$$- (\sigma_{a+b} + \cdots + \sigma_{a+1,b-1})$$

$$= \sigma_{a,b}.$$

For $r = 3$ and $\lambda = (a, b, c)$:

$$\det \begin{pmatrix} \sigma_a & \sigma_{a+1} & \sigma_{a+2} \\ \sigma_{b-1} & \sigma_b & \sigma_{b+1} \\ \sigma_{c-2} & \sigma_{c-1} & \sigma_c \end{pmatrix} = \sigma_a \, \sigma_{b,c} - \sigma_{b-1} \, \sigma_{a+1,c} + \sigma_{c-2} \, \sigma_{a+1,b+1}$$

$$= \ldots$$

$$= \sigma_{a,b,c}.$$

Back to the Motivating Problem

Question

Given four lines $\ell_1, \ldots, \ell_4$ in $\mathbb{P}^3$ in general position, how many lines intersect all four?

Now that we know Pieri's rule, we can compute

$$\sigma_1^4 = (\sigma_{2,0} + \sigma_{1,1})^2 = \sigma_{2,2} + \sigma_{2,2} = 2\sigma_{2,2}.$$

Hence the answer is 2. We will sketch another proof that does not use Schubert calculus.

Lemma

Given a point p and two lines ℓ_1, ℓ_2 in $\mathbb{P}^3$ in general position. Then there exists a unique line through p that intersect both ℓ_1 and ℓ_2.

For the proof, pick a general plane H and projects ℓ_1, ℓ_2 onto it from p. Connect p with the point of intersection on H to get the desired line.

Lines Intersecting Four Other Lines

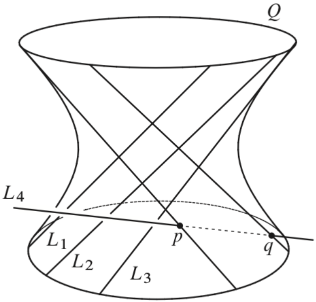

Figure: Figure 3.8 in "3264 and all that"

For each point $p \in \ell_3$, let M_p be the unique line passing through p and intersect both ℓ_1 and ℓ_2. Let $Q = \bigcup_p M_p$.

Key Fact

The lines M_p are disjoint, and Q is a quadric surface.

Thus, the fourth general line ℓ_4 will intersect Q at two points, corresponding to two lines M_p, M_q.

A Generalization

Question

Given four smooth curves $C_1, \ldots, C_4$ in $\mathbb{P}^3$ of degree $d_1, \ldots, d_4$ respectively and in general position, how many lines intersect all four curves?

The proof is similar to the previous problem. For a curve C in $\mathbb{P}^3$, let

$$\Gamma_C = \{\ell \in \mathrm{Gr}(2,4) \mid \ell \cap C \neq \emptyset\}.$$

We can show that this is a subvariety of $\mathrm{Gr}(2,4)$ of codimension 1, so $[\Gamma_C] = d\sigma_1$ for some integer d. To determine d, we use the *method of undertermined coefficients*. In other words, we multiply

$$[\Gamma_C]\sigma_{2,1} = d\sigma_1\sigma_{2,1} = d\sigma_{2,2}.$$

Hence d is the number of points in the intersection $\Gamma_C \cap \Omega_{2,1}(\mathcal{F})$, assuming it is transverse. This can be computed to be $\deg(C)$.

A Generalization

Question

Given four smooth curves $C_1, \ldots, C_4$ in $\mathbb{P}^3$ of degree $d_1, \ldots, d_4$ respectively and in general position, how many lines intersect all four curves?

Hence the intersection number is

$$\prod_i [\Gamma_{C_i}] = 2d_1 d_2 d_3 d_4.$$

Littlewood-Richardson Rule

For general partitions $\lambda, \mu \subset (k^{n-k})$, the product $\sigma_\lambda \cdot \sigma_\mu$ is a linear combination of $\{\sigma_\nu \mid |\nu| = |\lambda| + |\mu|\}$. In other words,

$$\sigma_\lambda \sigma_\mu = \sum_{|\nu|=|\lambda|+|nu|} c^\nu_{\lambda\mu} \sigma_\nu$$

for some integers $c^\nu_{\lambda\mu}$.

Question

Is there an algorithm/combinatorial formula to compute these coefficient?

The answer to this question is the *Littlewood-Richardson rule*, and requires us to take a detour into the world of symmetric polynomials and Young tableaux.

References

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