Schubert Calculus Day 1: The Schubert Stratification

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What is Enumerative Geometry?

Enumerative geometry is the study of counting algebro-geometric objects satisfying some given conditions.

- Given four lines in 3-dim space in general position, how many lines meet all four?
- Given a smooth cubic surface, how many lines does it contain?
- Given five plane conics in general position, how many conics are tangent to all five?

The first breakthrough in the subject was by Hermann Schubert (1848 - 1911). The ideas he introduced could be used to answer many enumerative questions about linear subspaces in projective space, and are now known as Schubert calculus.
Hilbert’s Fifteenth Problem

One of Hilbert’s 23 problems was to establish a rigorous foundation for Schubert calculus.

The problem consists in this: To establish rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert especially has determined on the basis of the so-called principle of special position, or conservation of number, by means of the enumerative calculus developed by him.

This problem, in the scope considered by Schubert, is now solved. However, there are many generalizations and questions still in need of answers.
Question

Given four lines $\ell_1, \ldots, \ell_4$ in $\mathbb{P}^3$ in general position, how many lines meet all four?

We could solve the problem by following these steps.

1. Define an object $\mathcal{G}$ that parametrizes lines in $\mathbb{P}^3$.
2. For each of the lines $\ell_i$, define a subset $\mathcal{G}_i$ of $\mathcal{G}$ parametrizing lines intersecting $\ell_i$.
3. Count the number of points in $\bigcap_{i=1}^{4} \mathcal{G}_i$.

In the rest of the lecture, we will make precise each of these steps.
The Grassmannian

Figure: Hermann Schubert (1848 - 1911)

Figure: Hermann Grassmann (1809 - 1877)
Let $V \cong \mathbb{C}^n$ be a $n$-dimensional vector space. The Grassmannian $\text{Gr}(k, V) = \text{Gr}(k, n)$ is the set of $k$-dimensional vector subspaces of $V$.

What is missing? More structure!

Pick a basis $e_1, \ldots, e_n$ of $V$. Then a $k$-dimensional subspace has as basis $k$ linearly independent vectors $\implies$ a $k \times n$ matrix of rank $k$.

Two sets of vectors generate the same subspace if they are $\text{GL}(k)$-equivalent.
Thus, we can write

$$\text{Gr}(k, n) = \{ k \times n \text{ matrices of rank } k \}/\text{GL}(k).$$

Let $U = \begin{pmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,n} \\ x_{2,1} & x_{2,2} & \ldots & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k,1} & x_{k,2} & \ldots & x_{k,n} \end{pmatrix}$.

Then $[U] \in \text{Gr}(k, n) \iff$ rows of $U$ are independent vectors $\iff$ some $k \times k$ minor of $U$ is not zero.
For $[U] \in \text{Gr}(k, n)$ and $J = \{j_1, \ldots, j_k\} \subseteq [n]$, denote by $U_J$ the square submatrix of $U$ with columns $j_1, \ldots, j_k$.

We define the Plücker embedding to be

$$\text{Gr}(k, n) \to \mathbb{P}^{\binom{n}{k} - 1}$$

$$[U] \mapsto [\det(U_J) \mid J \subseteq [n], |J| = k].$$

Check: this is well-defined (easy), and is actually an embedding (harder).

To prove the latter, we will use the coordinate-free version of the Plücker embedding.
The Plücker Embedding

Since $\Lambda^k V$ is a vector space of dimension $\binom{n}{k}$, we can consider its projectivization $\mathbb{P}(\Lambda^k V) \simeq \mathbb{P}(\binom{n}{k})^{-1}$.

The Plücker embedding can be described as follows: given a $k$-dim subspace $[U] \in \text{Gr}(k, n)$, pick a basis $v_1, \ldots, v_k$ and send it to the point corresponding to the line spanned by $\eta = v_1 \wedge \cdots \wedge v_k$ in $\mathbb{P}(\Lambda^k V)$.

Note that $v \in V$ satisfies $v \wedge \eta = 0$ iff $v$ is in the span of $v_1, \ldots, v_k$. Hence, different $k$-dim subspaces of $V$ are mapped to different points of $\mathbb{P}(\Lambda^k V)$, i.e. the map is one-to-one.
It remains to show that the image of \( \text{Gr}(k, n) \hookrightarrow \mathbb{P}(\Lambda^k V) \) is a closed subvariety. This is the locus of vectors \( \eta \in \Lambda^k V \) expressible as a wedge product \( v_1 \wedge \cdots \wedge v_k \) of \( k \) linearly independent vectors \( v_1, \ldots, v_k \in V \).

This happens if and only if the kernel of the multiplication map

\[
\varphi : V \xrightarrow{\wedge \eta} \Lambda^{k+1} V
\]

has dimension at least \( k \). Equivalently, this map has rank at most \( n - k \).

Writing \( \varphi \) in matrix form, this is the zero locus of the \((n - k + 1)\)-st minors, and hence an algebraic set.
Therefore, we have shown that

$$\text{Gr}(k, n) = \text{Proj} \mathbb{C}[p_J \mid J \subset [n], |J| = k]/I$$

for some homogeneous ideal $I$. This ideal is generated by quadratic polynomials in the Plücker coordinates known as \textit{Plücker relations}.

**Example**

For $\text{Gr}(2, 4) \hookrightarrow \text{Proj} \mathbb{C}[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}]$, we only have one Plücker relation $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$.

In other words, $\text{Gr}(2, 4) \subset \mathbb{P}^5$ is a smooth quadric.
Affine Charts

Given \( J = \{j_1, \ldots, j_k\} \subset [n] \), we can consider the open subset of \( \text{Gr}(k, n) \) where the \( J \)-th Plücker coordinate does not vanish. The \( k \times k \) submatrix corresponding to \( J \) can be multiplied using the \( \text{GL}(k) \) action to be the identity matrix. Thus, the open subset has the form

\[
\begin{pmatrix}
* & 1 & * & * & 0 & 0 & *\\
* & 0 & * & * & 1 & 0 & *\\
* & 0 & * & * & 0 & 1 & *
\end{pmatrix}
\]

and is isomorphic to \( \mathbb{A}^{k(n-k)} \).

Theorem

The Grassmannian \( \text{Gr}(k, n) \) is a projective variety locally isomorphic to affine space \( \mathbb{A}^{k(n-k)} \). It is also irreducible and smooth.
Schubert Cells

Fix a basis \(e_1, \ldots, e_n\) and consider the complete flag

\[ F : \langle e_1 \rangle = F_1 \subset \langle e_1, e_2 \rangle = F_2 \subset \cdots \subset \langle e_1, e_2, \ldots, e_n \rangle = F_n = \mathbb{C}^n. \]

**Lemma**

Every subspace in \(\text{Gr}(k, n)\) can be represented by a unique \(k \times n\) matrix in row echelon form.

**Example**

Consider \(U = \text{span}\{6e_1 + 3e_2, 4e_1 + 2e_3, 9e_1 + e_3 + e_4\} \in G(3, 4)\). In matrix form, this is

\[
\begin{pmatrix}
6 & 3 & 0 & 0 \\
4 & 0 & 2 & 0 \\
9 & 0 & 1 & 1
\end{pmatrix} =
\begin{pmatrix}
3 & 0 & 0 \\
0 & 2 & 0 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
2 & 1 & 0 & 0 \\
2 & 0 & 1 & 0 \\
7 & 0 & 0 & 1
\end{pmatrix}.
\]
For $[U] \in \text{Gr}(k, n)$ with $U$ in canonical form, the columns of the leading 1’s determine a subset of size $k$ in $[n]$. This determines the position of $U$ with respect to the fixed basis.

**Example**

In $\text{Gr}(4, 10)$, all subspaces with position $\{2, 4, 7, 9\}$ has the following form

\[
\begin{pmatrix}
* & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & 0 & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & 0 & * & 0 & * & * & 1 & 0 & 0 & 0 \\
* & 0 & * & 0 & * & * & 0 & * & 1 & 0
\end{pmatrix}.
\]

Note that the framed zeros form a (rotated) Young diagram. In fact, the set of positions correspond to the set of Young diagrams contained in the $k \times (n - k)$ rectangle.
Definition

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \subset (k^{n-k})$ (which corresponds to a position $P$), the Schubert cell corresponding to $\lambda$ is

$$\Omega^\circ_\lambda = \{ [U] \in \text{Gr}(k, n) \mid \text{position}(U) = P \}.$$

Definition

The Schubert variety corresponding to $\lambda$ is $\Omega_\lambda = \text{the closure of } \Omega^\circ_\lambda \text{ in the Zariski topology of Gr}(k, n)$.

When we take the closure, we can shift the leading 1’s to the left of the given position.
Schubert Cells

In terms of the complete flag \( F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n \), the condition on the position can be translated as

\[
\Omega_\lambda = \{ [U] \in \operatorname{Gr}(k, n) \mid \dim(U \cap F_{n-k+i-\lambda}) \geq i \text{ for all } 1 \leq i \leq k \},
\]

and

\[
\Omega^\circ_\lambda = \{ \dim(U \cap F_j) = i \text{ for all } n-k+i-\lambda_j \leq j \leq n-k+i-\lambda_{j+1} \}.
\]

Note that \( \Omega^\circ_\lambda \cong \mathbb{A}^{k(n-k)-|\lambda|} \). When \( \lambda = k(n-k) \), we have \( \Omega_\lambda = \{ \ast \} \) and when \( \lambda = (0) \), we have \( \Omega_\lambda = \operatorname{Gr}(k, n) \).
What is the boundary $\Omega_\lambda \setminus \Omega_\lambda^\circ$?

Example

Consider $\text{Gr}(4, 10)$ with $\lambda = (5, 4, 2, 1)$.

\[
\begin{pmatrix}
* & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & 0 & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & 0 & * & 0 & * & * & 1 & 0 & 0 & 0 \\
* & 0 & * & 0 & * & * & 0 & * & 1 & 0 \\
\end{pmatrix}.
\]

Theorem

Let $\lambda \subset (k^{n-k})$. Then we have $\Omega_\lambda = \bigcup_{\mu \supset \lambda} \Omega_\mu^\circ$. 
Taking $\lambda = (0)$, we get the *Schubert stratification*

$$\text{Gr}(k, n) = \bigsqcup_{\lambda \subseteq (k^{n-k})} \Omega^\circ_{\lambda}.$$ 

This is an affine stratification in the sense that
- Each of the cell is isomorphic to affine space.
- Each cell’s closure is the union of other cells.

**Example**

For $\text{Gr}(1, n + 1) = \mathbb{P}^n$, the Schubert stratification corresponds to

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \mathbb{A}^{n-2} \sqcup \cdots \sqcup \mathbb{A}^1 \sqcup \{\ast\}.$$
Since we are working over $\mathbb{C}$, the Schubert stratification gives a CW complex structure for $\text{Gr}(k, n)$ with cells in even (real) dimension. Thus, the singular cohomology ring $H^*(\text{Gr}(k, n))$ is generated by the class $\sigma_\lambda$ of the Schubert varieties $\Omega_\lambda$.

(The same statement holds if we consider $\text{Gr}(k, n)$ as an algebraic variety and take the Chow ring $A^*(\text{Gr}(k, n))$.)

Note that as Schubert classes, the choice of complete flags is not important as they are all related by a $\text{GL}(k)$ action. Hence we can define the Schubert variety $\Omega_\lambda(\mathcal{F})$ with respect to any complete flag $\mathcal{F}$. 

Examples of Schubert Varieties

We give more examples of Schubert varieties/cells and how to interpret them.

Example

The set of $k$-subspaces $U$ meeting a given space $F_\ell$ of dimension $\ell$ nontrivially is

$$\Omega_{n-k+1-\ell}(\mathcal{F}) = \{[U] \mid U \cap F_\ell \neq 0\}.$$  

Example

The set of $k$-subspaces $U$ contained in a given $\ell$-subspace $F_\ell$ is

$$\Omega_{(n-\ell)k}(\mathcal{F}) = \{[U] \mid U \subset F_\ell\}.$$  

The set of $k$-subspaces $U$ containing in a given $r$-subspace $F_r$ is

$$\Omega_{(n-k)r}(\mathcal{F}) = \{[U] \mid U \supset F_r\}.$$
Re-interpreting the Question

Recall our motivating question

Question

Given four lines \( \ell_1, \ldots, \ell_4 \) in \( \mathbb{P}^3 \) in general position, how many lines meet all four?

We proceed to turn this into a question about the Grassmannian as follows.

1. The set of lines in \( \mathbb{P}^3 \) is the same as the set of 2-planes in 4-space. Hence we consider \( \text{Gr}(2, 4) \).
2. The Schubert variety corresponding to the lines that intersect a given line \([\ell_i]\) is \( \Omega_1(\mathcal{F}) \) with \( F_2 = \ell_i \).
3. To get the number \( c \) of lines meeting all of \( \ell_1, \ldots, \ell_4 \), we proceed to compute the cup product \( \sigma_{1}^{4} = c \cdot \sigma_{2,2} \).
Next time

- What is the ring structure of $H^*(Gr(k, n))$?

- Connection with symmetric polynomials.

David Eisenbud and Joe Harris. 3264 and all that: A second course in algebraic geometry. Cambridge University Press, 2016.
