Schubert Calculus Day 1: The Schubert Stratification

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What is Enumerative Geometry?

Enumerative geometry is the study of counting algebro-geometric objects satisfying some given conditions.

- Given four lines in 3-dim space in general position, how many lines meet all four?
- Given a smooth cubic surface, how many lines does it contain?
- Given five plane conics in general position, how many conics are tangent to all five?

The first breakthrough in the subject was by Hermann Schubert (1848 - 1911). The ideas he introduced could be used to answer many enumerative questions about linear subspaces in projective space, and are now known as *Schubert calculus*.

Hilbert's Fifteenth Problem

One of Hilbert's 23 problems was to establish a rigorous foundation for Schubert calculus.

> *The problem consists in this: To establish rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert especially has determined on the basis of the so-called principle of special position, or conservation of number, by means of the enumerative calculus developed by him.*

This problem, in the scope considered by Schubert, is now solved. However, there are many generalizations and questions still in need of answers.

Strategy of Proof

Question

Given four lines $\ell_1, \ldots, \ell_4$ in $\mathbb{P}^3$ in general position, how many lines meet all four?

We could solve the problem by following these steps.

1. Define an object $\mathbb{G}$ that parametrizes lines in $\mathbb{P}^3$.
2. For each of the lines ℓ_i, define a subset $\mathbb{G}_i$ of $\mathbb{G}$ parametrizing lines intersecting ℓ_i.
3. Count the number of points in $\bigcap_{i=1}^4 \mathbb{G}_i$.

In the rest of the lecture, we will make precise each of these steps.

The Grassmannian

Figure: Hermann Schubert (1848 - 1911)

Figure: Hermann Grassmann (1809 - 1877)

The Grassmannian

"Definition"

Let $V \simeq \mathbb{C}^n$ be a n-dimensional vector space. The Grassmannian $\mathrm{Gr}(k, V) = \mathrm{Gr}(k, n)$ is the set of k-dimensional vector subspaces of V.

What is missing? More structure!

Pick a basis $e_1, \dots, e_n$ of V. Then a k-dimensional subspace has as basis k linearly independent vectors $\implies$ a $k \times n$ matrix of rank k.

Two sets of vectors generate the same subspace if they are $\mathrm{GL}(k)$-equivalent.

The Grassmannian

Thus, we can write

$$\mathrm{Gr}(k,n) = \{k \times n \text{ matrices of rank } k\}/\,\mathrm{GL}(k).$$

Let $U = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k,1} & x_{k,2} & \cdots & x_{k,n} \end{pmatrix}$.

Then $[U] \in \mathrm{Gr}(k,n) \iff$ rows of U are independent vectors $\iff$ some $k \times k$ minor of U is not zero.

The Plücker Embedding

For $[U] \in \mathrm{Gr}(k, n)$ and $J = \{j_1, \ldots, j_k\} \subset [n]$, denote by U_J the square submatrix of U with columns $j_1, \ldots, j_k$.

We define the Plücker embedding to be

$$\begin{aligned} \mathrm{Gr}(k, n) &\to \mathbb{P}^{\binom{n}{k}-1} \\ [U] &\mapsto [\det(U_J) \mid J \subset [n], |J| = k]. \end{aligned}$$

Check: this is well-defined (easy), and is actually an embedding (harder).

To prove the latter, we will use the coordinate-free version of the Plücker embedding.

The Plücker Embedding

Since $\Lambda^k V$ is a vector space of dimension $\binom{n}{k}$, we can consider its projectivization $\mathbb{P}(\Lambda^k V) \simeq \mathbb{P}^{\binom{n}{k}-1}$.

The Plücker embedding can be described as follows: given a k-dim subspace $[U] \in \mathrm{Gr}(k, n)$, pick a basis $v_1, \ldots, v_k$ and send it to the point corresponding to the line spanned by $\eta = v_1 \wedge \cdots \wedge v_k$ in $\mathbb{P}(\Lambda^k V)$.

Note that $v \in V$ satisfies $v \wedge \eta = 0$ iff v is in the span of $v_1, \ldots, v_k$. Hence, different k-dim subspaces of V are mapped to different points of $\mathbb{P}(\Lambda^k V)$, i.e. the map is one-to-one.

The Plücker Embedding

It remains to show that the image of $\mathrm{Gr}(k,n) \hookrightarrow \mathbb{P}(\Lambda^k V)$ is a closed subvariety. This is the locus of vectors $\eta \in \Lambda^k V$ expressible as a wedge product $v_1 \wedge \cdots \wedge v_k$ of k linearly independent vectors $v_1, \ldots, v_k \in V$.

This happens if and only if the kernel of the multiplication map

$$\varphi : V \xrightarrow{\wedge \eta} \Lambda^{k+1} V$$

has dimension at least k. Equivalently, this map has rank at most $n-k$.

Writing φ in matrix form, this is the zero locus of the $(n-k+1)$-st minors, and hence an algebraic set.

The Plücker Embedding

Therefore, we have shown that

$$\mathrm{Gr}(k,n) = \operatorname{Proj} \mathbb{C}[p_J \mid J \subset [n], |J| = k]/I$$

for some homogeneous ideal I. This ideal is generated by quadratic polynomials in the Plücker coordinates known as *Plücker relations*.

Example

For $\mathrm{Gr}(2,4) \hookrightarrow \operatorname{Proj} \mathbb{C}[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}]$, we only have one Plücker relation $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$.
In other words, $\mathrm{Gr}(2,4) \subset \mathbb{P}^5$ is a smooth quadric.

Affine Charts

Given $J = \{j_1, \ldots, j_k\} \subset [n]$, we can consider the open subset of $\mathrm{Gr}(k, n)$ where the J-th Plücker coordinate does not vanish. The $k \times k$ submatrix corresponding to J can be multiplied using the $\mathrm{GL}(k)$ action to be the identity matrix. Thus, the open subset has the form

$$\begin{pmatrix} * & 1 & * & * & 0 & 0 & * \\ * & 0 & * & * & 1 & 0 & * \\ * & 0 & * & * & 0 & 1 & * \end{pmatrix}$$

and is isomorphic to $\mathbb{A}^{k(n-k)}$.

Theorem

The Grassmannian $\mathrm{Gr}(k, n)$ is a projective variety locally isomorphic to affine space $\mathbb{A}^{k(n-k)}$. It is also irreducible and smooth.

Schubert Cells

Fix a basis $e_1, \dots, e_n$ and consider the complete flag
$\mathcal{F} : \langle e_1 \rangle = F_1 \subset \langle e_1, e_2 \rangle = F_2 \subset \cdots \subset \langle e_1, e_2, \dots, e_n \rangle = F_n = \mathbb{C}^n$.

Lemma

Every subspace in $\mathrm{Gr}(k, n)$ can be represented by a unique $k \times n$ matrix in row echelon form.

Example

Consider $U = \mathrm{span}\langle 6e_1 + 3e_2, 4e_1 + 2e_3, 9e_1 + e_3 + e_4 \rangle \in G(3, 4)$. In matrix form, this is

$$\begin{pmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \end{pmatrix}.$$

Schubert Cells

For $[U] \in \mathrm{Gr}(k, n)$ with U in canonical form, the columns of the leading 1's determine a subset of size k in $[n]$. This determines the *position* of U with respect to the fixed basis.

Example

In $\mathrm{Gr}(4, 10)$, all subspaces with position $\{2, 4, 7, 9\}$ has the following form

$$\begin{pmatrix} * & 1 & \boxed{0} & 0 & \boxed{0} & \boxed{0} & 0 & \boxed{0} & 0 & \boxed{0} \\ * & 0 & * & 1 & \boxed{0} & \boxed{0} & 0 & \boxed{0} & 0 & \boxed{0} \\ * & 0 & * & 0 & * & * & 1 & \boxed{0} & 0 & \boxed{0} \\ * & 0 & * & 0 & * & * & 0 & * & 1 & \boxed{0} \end{pmatrix}.$$

Note that the framed zeros form a (rotated) Young diagram. In fact, the set of positions correspond to the set of Young diagrams contained in the $k \times (n - k)$ rectangle.

Schubert Cells

Definition

Given a partition $\lambda = (\lambda_1, \dots, \lambda_k) \subset (k^{n-k})$ (which corresponds to a position P), the *Schubert cell* corresponding to λ is

$$\Omega_\lambda^\circ = \{[U] \in \mathrm{Gr}(k,n) \mid \mathrm{position}(U) = P\}.$$

Definition

The *Schubert variety* corresponding to λ is $\Omega_\lambda =$ the closure of Ω_λ° in the Zariski topology of $\mathrm{Gr}(k,n)$.

When we take the closure, we can shift the leading 1's to the left of the given position.

Schubert Cells

In terms of the complete flag $F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n$, the condition on the position can be translated as

$$\Omega_\lambda = \{[U] \in \mathrm{Gr}(k,n) \mid \dim(U \cap F_{n-k+i-\lambda_i}) \geqslant i \text{ for all } 1 \leqslant i \leqslant k\},$$

and

$$\Omega_\lambda^\circ = \{\dim(U \cap F_j) = i \text{ for all } n-k+i-\lambda_i \leqslant j \leqslant n-k+i-\lambda_{i+1}\}.$$

Note that $\Omega_\lambda^\circ \simeq \mathbb{A}^{k(n-k)-|\lambda|}$. When $\lambda = k(n-k)$, we have $\Omega_\lambda = \{*\}$ and when $\lambda = (0)$, we have $\Omega_\lambda = \mathrm{Gr}(k,n)$.

Schubert Stratification

What is the boundary $\Omega_\lambda \setminus \Omega_\lambda^\circ$?

Example

Consider $\mathrm{Gr}(4,10)$ with $\lambda = (5,4,2,1)$.

$$\begin{pmatrix} * & 1 & \boxed{0} & 0 & \boxed{0} & \boxed{0} & 0 & \boxed{0} & 0 & \boxed{0} \\ * & 0 & * & 1 & \boxed{0} & \boxed{0} & 0 & \boxed{0} & 0 & \boxed{0} \\ * & 0 & * & 0 & * & * & 1 & \boxed{0} & 0 & \boxed{0} \\ * & 0 & * & 0 & * & * & 0 & * & 1 & \boxed{0} \end{pmatrix}.$$

Theorem

Let $\lambda \subset (k^{n-k})$. Then we have $\Omega_\lambda = \bigsqcup_{\mu \supset \lambda} \Omega_\mu^\circ$.

Schubert Stratification

Taking $\lambda = (0)$, we get the *Schubert stratification*

$$\mathrm{Gr}(k, n) = \bigsqcup_{\lambda \subset (k^{n-k})} \Omega^\circ_\lambda.$$

This is an affine stratification in the sense that

- Each of the cell is isomorphic to affine space.
- Each cell's closure is the union of other cells.

Example

For $\mathrm{Gr}(1, n+1) = \mathbb{P}^n$, the Schubert stratification corresponds to

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \mathbb{A}^{n-2} \sqcup \cdots \sqcup \mathbb{A}^1 \sqcup \{*\}.$$

Cohomology Ring

Since we are working over $\mathbb{C}$, the Schubert stratification gives a CW complex structure for $\mathrm{Gr}(k, n)$ with cells in even (real) dimension. Thus, the singular cohomology ring $H^*(\mathrm{Gr}(k, n))$ is generated by the class σ_λ of the Schubert varieties Ω_λ.

(The same statement holds if we consider $\mathrm{Gr}(k, n)$ as an algebraic variety and take the Chow ring $A^*(\mathrm{Gr}(k, n))$.)

Note that as Schubert classes, the choice of complete flags is not important as they are all related by a $\mathrm{GL}(k)$ action. Hence we can define the Schubert variety $\Omega_\lambda(\mathcal{F})$ with respect to any complete flag $\mathcal{F}$.

Examples of Schubert Varieties

We give more examples of Schubert varieties/cells and how to interpret them.

Example

The set of k-subspaces U meeting a given space F_l of dimension ℓ nontrivially is

$$\Omega_{n-k+1-\ell}(\mathcal{F}) = \{[U] \mid U \cap F_l \neq 0\}.$$

Example

The set of k-subspaces U contained in a given ℓ-subspace F_l is

$$\Omega_{(n-\ell)^k}(\mathcal{F}) = \{[U] \mid U \subset F_\ell\}.$$

The set of k-subspaces U containing in a given r-subspace F_r is

$$\Omega_{(n-k)^r}(\mathcal{F}) = \{[U] \mid U \supset F_r\}.$$

Re-interpreting the Question

Recall our motivating question

Question

Given four lines $\ell_1, \ldots, \ell_4$ in $\mathbb{P}^3$ in general position, how many lines meet all four?

We proceed to turn this into a question about the Grassmannian as follows.

1. The set of lines in $\mathbb{P}^3$ is the same as the set of 2-planes in 4-space. Hence we consider $\mathrm{Gr}(2, 4)$.
2. The Schubert variety corresponding to the lines that intersect a given line $[\ell_i]$ is $\Omega_1(\mathcal{F})$ with $F_2 = \ell_i$.
3. To get the number c of lines meeting all of $\ell_1, \ldots, \ell_4$, we proceed to compute the cup product $\sigma_1^4 = c \cdot \sigma_{2,2}$.

Next time

- What is the ring structure of $H^*(\mathrm{Gr}(k,n))$?
- Connection with symmetric polynomials.

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